

Goodness-of-fit test for randomly censored data based on maximum correlation

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Abstract

In this paper we study a goodness-of-fit test based on the maximum correlation coefficient, in the context of randomly censored data. We construct a new test statistic under general right-censoring and prove its asymptotic properties. Additionally, we study a special case, when the censoring mechanism follows the well-known Koziol-Green model. We present an extensive simulation study on the empirical power of these two versions of the test statistic, showing their advantages over the widely used Pearson-type test. Finally, we apply our test to the head-and-neck cancer data.

MSC: 62N01, 62N03, 62G10, 62G20.

Keywords: Goodness-of-fit, Kaplan-Meier estimator, maximum correlation, random censoring.

1. Introduction

In many medical studies one encounters data which are not fully observed but censored from the right. For example, in the head-and-neck cancer trial studied by Nikulin and Haghghi (2006), one observes survival times for 42 out of 51 patients, whereas for the remaining 9 patients only the time to follow-up is given. Let Y_1, \dots, Y_n be the lifetimes of interest, e.g., the survival times of head-and-neck cancer patients, coming from a continuous distribution function F and let C_1, \dots, C_n be the censoring times (that is, the times to follow-up) coming from a distribution function G . In the context of right-censored data, for every $i = 1, \dots, n$, we observe

$$X_i = \min(Y_i, C_i) \text{ and } \delta_i = 1_{\{Y_i \leq C_i\}},$$

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where 1_A denotes the indicator function, being equal to 1 if A is fulfilled and 0 otherwise. The unknown distribution function of the lifetimes F can be estimated by the well-known product-limit estimator introduced by Kaplan and Meier (1958). However, if the shape of the distribution could be assumed, there would be a substantial gain in the efficiency of statistical procedures. For instance, in the example of head-and-neck cancer data, Nikulin and Haghighi (2006) suggest that the lifetimes follow the Generalized-Power Weibull family and that hypothesis is tested. Therefore, goodness-of-fit tests are an important statistical tool when dealing with (right-)censored data. Under complete data set-up we have a multitude of goodness-of-fit tests to select from. See, e.g., Darling (1957) or Massey (1951) for the historical literature on the subject and Torabi et al. (2016) or Novoa-Muñoz and Jiménez-Gamero (2016), among many others, for the most recent publications. Some widely used tests for complete data, like Kolmogorov-Smirnov or Cramer-von Mises, are difficult to apply in the presence of censoring, since the limit distribution depends on the censoring distribution G . See Balakrishnan et al. (2015) for a recent overview on this kind of tests with randomly censored data. Other classical approaches are Koziol and Green (1976) and Akritas (1988). The former is more restrictive, since it is based on the assumption that the distribution function G follows the so called Koziol-Green model, whereas the latter is a χ^2 test applied to general random censoring. This is the reason why the Pearson-type goodness-of-fit test proposed by Akritas (1988) is so far the best option for randomly censored data with unknown censoring distribution. Nevertheless, it requires a partition of the observations into cells jointly with an adequate choice of number of classes, since the power of the test may vary depending on the degrees of freedom. In this work we propose a new goodness-of-fit test based on the maximum correlation coefficient, with normal limiting distribution and, therefore, straightforward to apply.

We start by introducing the maximum correlation in a more general set-up. Let Y_1 and Y_2 be two random variables with finite second order moments, joint cumulative distribution function (cdf) H and marginals F_1 and F_2 , respectively. The Hoeffding representation of the correlation coefficient is given by

$$\rho(F_1, F_2) = \frac{1}{\sigma_1 \sigma_2} \int_{\mathbb{R}^2} (H(x, y) - F_1(x)F_2(y)) dx dy,$$

where σ_i denotes the standard deviation of Y_i . Furthermore, the maximum correlation of the pair of random variables (Y_1, Y_2) is defined as the correlation coefficient $\rho^+(F_1, F_2)$ corresponding to the bivariate cdf $H^+(x, y) = \min(F_1(x), F_2(y))$, the upper Fréchet bound of $H(x, y)$. The cdf $H^+(x, y)$ is a singular distribution, having support on the one-dimensional set $\{(x, y) \in \mathbb{R}^2 : F_1(x) = F_2(y)\}$, and the maximum correlation coefficient is given by

$$\rho^+(F_1, F_2) = \frac{1}{\sigma_1 \sigma_2} \left(\int_0^1 F_1^{-1}(p) F_2^{-1}(p) dp - \mu_1 \mu_2 \right), \quad (1)$$

where F_i^{-1} is the inverse of F_i and μ_i is the mean of Y_i . This maximum correlation, $\rho^+(F_1, F_2)$, is a measure of agreement between F_1 and F_2 , since $\rho^+ = 1$ if and only if $F_1 = F_2$ up to a scale and location change. In particular, Cuadras and Fortiana (1993) proposed the statistic based on $\rho^+(F, F_0)$ as a measure of goodness of fit of an iid sample Y_1, \dots, Y_n with cdf F , to a given distribution F_0 . The goodness-of-fit test based on maximum correlation was further studied by Fortiana and Grané (2003), Grané (2012) and Grané and Tchirina (2013).

As in the latter publications, the present paper is devoted to testing uniformity, i.e. $F_0 = F_U$, a $[0, 1]$ uniform distribution. As shown by Fortiana and Grané (2003) the asymptotic approximation of $\rho^+(F, F_U)$ is available, but convergence to its limiting law is rather slow. This led to defining

$$Q = \frac{\sigma}{\sqrt{1/12}} \rho^+(F, F_U) = 6 \int_0^1 x(2F(x) - 1)F(dx), \quad (2)$$

where σ is the standard deviation of $Y_i \sim F$, which equals one if $F = F_U$.

The goal of this paper is to study a test statistic based on Q when Y_1, \dots, Y_n may not be fully observed but censored from the right by censoring times C_1, \dots, C_n . More precisely, we wish to test the hypothesis $H_0 : F = F_U$, where F_U is the cdf of a $[0, 1]$ uniform random variable, based on the sample $(X_i, \delta_i)_{i=1, \dots, n}$, where $X_i = \min(Y_i, C_i)$, with $X_i \in [0, 1]$. Nevertheless, our approach is not restricted to testing uniformity. We can also consider a more general null hypothesis F_0 , since the transformed random variable $F_0(Y)$ follows a $[0, 1]$ uniform distribution under $H_0 : F = F_0$. That is, $\tilde{Y} = F_0(Y) \sim F_U$ under the null hypothesis. Then, setting $\tilde{C} = F_0(C)$ and since $\{\tilde{Y}_i \leq \tilde{C}_i\} = \{Y_i \leq C_i\}$, leads us to testing uniformity based on the iid sample $(\tilde{X}_1, \delta_1), \dots, (\tilde{X}_n, \delta_n)$, where

$$\tilde{X}_i = \min(\tilde{Y}_i, \tilde{C}_i) \quad \text{and} \quad \delta_i = 1_{\{\tilde{Y}_i \leq \tilde{C}_i\}}.$$

Hence, testing for uniformity is equivalent to testing for a fully specified continuous distribution. Even though it seems that we could extend the work of Fortiana and Grané (2003) by setting $Q_n = 6 \int_0^1 x(2F_n(x) - 1)F_n(dx)$, where F_n denotes the Kaplan-Meier estimator for censored data, it is far from being true. In contrast to the empirical distribution under completely observed data, the Kaplan-Meier estimator is biased (see Stute (1994), for details). In Section 2 we show that such a plug-in estimator suffers from the bias of the product-limit estimator and, therefore, $E(Q_n) = 1$ does not hold under H_0 . To avoid this problem we propose to re-write Q in such a way that it can be estimated by U-statistics. This leads to significant bias (and variance) reduction. In Section 3 we prove the asymptotic normality of the proposed estimator and in Section 4 we present our new goodness-of-fit test. In Section 5 we present an extensive simulation study. Finally, in Section 6 we adapt the test statistic to the case of composite null hypothesis and apply our test to the head-and-neck cancer data from Nikulin and Haghghi (2006).

2. Test statistic

In this section we propose our new goodness-of-fit statistic for randomly censored data, based on the modified maximum correlation coefficient. Recall that, under $H_0 : F = F_U$, the quantity

$$Q = \frac{\sigma}{\sqrt{1/12}} \rho^+(F, F_U) = 6 \int_0^1 x(2F(x) - 1)F(dx)$$

equals one. Hence in the following we prefer to work with

$$Q^1 = Q - 1 = 6 \int_0^1 x(2F(x) - 1)F(dx) - 1 \quad (3)$$

which equals zero if H_0 is true.

First, we define a plug-in estimator of Q^1 by replacing F in (3) with the well-known Kaplan-Meier estimator. We obtain

$$Q_n^1 = 6 \int_0^1 x(2F_n(x) - 1)F_n(dx) - 1, \quad (4)$$

where F_n is defined as follows

$$F_n(x) = 1 - \prod_{\bar{x}_i \leq x} \left[1 - \frac{\delta_i}{\sum_{k=1}^n 1_{\{X_k \geq \bar{x}_i\}}} \right]. \quad (5)$$

It turns out that, under the null hypothesis and for finite samples, the plug-in estimator Q_n^1 suffers from significant bias and its convergence to the limiting distribution is very slow.

To solve this problem, we propose to estimate Q^1 with a U-statistic. For this, note that if F is a continuous cdf and $\text{supp}(F) \subseteq [0, 1]$, then

$$2 \int_0^1 F(x)F(dx) = 1.$$

Hence

$$\begin{aligned} Q^1 &= \int_0^1 (6x(2F(x) - 1) - 2F(x))F(dx) = \int_0^1 [(6x - 2)F(x) - 6x(1 - F(x))]F(dx) \\ &= \int_0^1 \int_0^1 [(6x - 2)1_{\{y \leq x\}} - 6x1_{\{y > x\}}]F(dx)F(dy). \end{aligned} \quad (6)$$

Now we may replace the unknown quantities by their estimators. For this we introduce the jumps of the Kaplan-Meier estimator by setting

$$w_{in} = F_n(X_i) - F_n(X_{i-}),$$

where $F_n(x-)$ is the left-continuous version of $F_n(x)$, which is defined analogously as (5) but with the product over all $X_i < x$.

Finally, the estimator of Q^1 is given by

$$\tilde{Q}_n = \sum_{i=1}^n \sum_{j \neq i} w_{in} w_{jn} h(X_i, X_j), \tag{7}$$

where

$$h(x_1, x_2) = (6x_1 - 2)1_{\{x_2 \leq x_1\}} - 6x_1 1_{\{x_2 > x_1\}}.$$

To illustrate the advantages of using \tilde{Q}_n over the plug-in estimator Q_n^1 , in panel (a) of Figure 1, we present the bias and variance of those estimators under the null hypothesis, that is, when the data come from the $[0, 1]$ uniform distribution. Additionally, in panels (b)-(c) of Figure 1, we compare the kernel density estimators of the standardized versions of \tilde{Q}_n and Q_n^1 to that of the standard normal distribution. The standardization is done using the estimated asymptotic variances, discussed later on. Clearly, the U-statistic \tilde{Q}_n exhibits much smaller bias (and variance) than Q_n^1 and, additionally, its standardized version fits nicely the standard normal distribution for all the considered censoring rates.

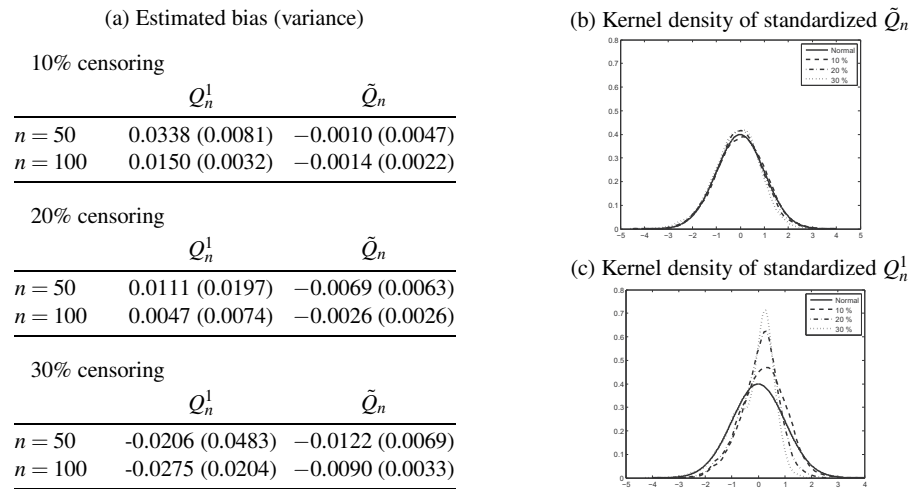


Figure 1: Comparison between \tilde{Q}_n and the plug-in estimator Q_n^1 : Estimated bias (variance) based on 5000 trials and kernel densities for $n = 200$.

3. Asymptotic properties

In this section we study the asymptotic properties of our test statistic \tilde{Q}_n . Firstly, we consider \tilde{Q}_n under a general censoring mechanism, that is, without assuming any shape for the distribution function of the censoring times $G(x) = P(C \leq x)$. Secondly, we apply the results to the special case of the Koziol-Green model. Recall that $F(x) = P(Y \leq x)$ is the cdf of the lifetimes of interest. We need the following assumptions A1-A2, which assure that the asymptotic variance is bounded and censoring is not too heavy. These conditions allow us to apply the limit theorems from Stute (1995) in order to prove the asymptotic normality:

$$A1: \int_0^1 \frac{F(du)}{1-G(u)} < \infty$$

$$A2: \int_0^1 |\varphi(u)|C^{1/2}(u)F(du) < \infty$$

where $\varphi(x) = 12xF(x) - 6x - 2 - 12 \int_0^x yF(dy) + 6 \int_0^1 yF(dy)$ is a score function, $C(x) = \int_0^x \frac{G(dy)}{(1-G(y))^2(1-F(y))}$ and F is continuous with support in $[0, 1]$.

Theorem 1 Under A1 and A2, we have

$$\sqrt{n}(\tilde{Q}_n - Q^1) \rightarrow \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \int_0^1 \frac{\varphi^2(x)}{1-G(x)}F(dx) - \left[\int_0^1 \varphi(x)F(dx) \right]^2 - \int_0^1 \left[\int_x^1 \varphi(y)F(dy) \right]^2 \frac{(1-F(x))G(dx)}{(1-H(x))^2}$$

and

$$\varphi(x) = 12xF(x) - 6x - 2 - 12 \int_0^x yF(dy) + 6 \int_0^1 yF(dy).$$

Proof. See Appendix.

Consequently, we have that

Corollary 1 Under H_0 , A1 and A2, we have

$$\sqrt{n}\tilde{Q}_n \rightarrow \mathcal{N}(0, \sigma^2).$$

The variance under H_0 would not simplify, since it does depend on the distribution function of the censoring times G , which is unknown. Nevertheless, under the Koziol-

Green model, we have an explicit expression for σ^2 . First, recall that G follows a Koziol-Green model if

$$1 - G(x) = (1 - F(x))^\beta,$$

where $\beta > 0$ is an unknown parameter. However, we can see that

$$p := P(Y > C) = \frac{\beta}{\beta + 1} \text{ and } 1 - p = \int (1 - G(x))F(dx).$$

Hence β can be easily estimated using Kaplan-Meier estimators for F and G . Finally, it is easy to check that assumptions A1 and A2 are fulfilled under the Koziol-Green model with $\beta \in (0, 1)$, that is, if the censoring is not heavier than 50%, which is a very reasonable assumption. So, as a consequence of Corollary 1, we get the following result.

Corollary 2 *Under the Koziol-Green model with $\beta \in (0, 1)$ we have that, under H_0 ,*

$$\sqrt{n}\tilde{Q}_n \rightarrow \mathcal{N}(0, \sigma_{KG}^2),$$

where

$$\sigma_{KG}^2 = \frac{-\beta^4 + 4\beta^3 - 17\beta^2 + 38\beta - 24}{(\beta - 1)(\beta - 2)(\beta - 3)(\beta - 4)(\beta - 5)}.$$

4. Goodness-of-fit test

Once the test statistic is proposed and its limiting distribution is established, we are in the position to define the goodness-of-fit test. For this we estimate the asymptotic variance σ^2 using the plug-in principle, that is, by replacing the unknown quantities with their estimators. First, we define the distribution function of the observed times $\tilde{H}(x) = P(X \leq x)$ and set $\tilde{H}_n(x) = \frac{1}{n} \sum_{i=1}^n 1_{\{X_i \leq x\}}$ as its empirical counterpart. Moreover, let

$$H^0(x) = \mathbb{P}(X \leq x, \delta = 0) = \int_0^x (1 - F(u))G(du)$$

and

$$H^1(x) = \mathbb{P}(X \leq x, \delta = 1) = \int_0^x (1 - G(u))F(du)$$

be the subdistributions related to the observed censored and uncensored lifetimes. Their estimators are defined as follows

$$H_n^0(x) = \frac{1}{n} \sum_{i=1}^n 1_{\{X_i \leq x\}} (1 - \delta_i)$$

and

$$H_n^1(x) = \frac{1}{n} \sum_{i=1}^n 1_{\{X_i \leq x\}} \delta_i.$$

Hence

$$\begin{aligned} \sigma_n^2 = & \frac{1}{n} \sum_{i=1}^n \frac{\varphi_n^2(X_i)}{(1 - G_n(X_{i-}))^2} \delta_i - \left[\frac{1}{n} \sum_{i=1}^n \frac{\varphi_n(X_i)}{1 - G_n(X_{i-})} \delta_i \right]^2 \\ & - \frac{1}{n} \sum_{i=1}^n \frac{1 - \delta_i}{(1 - H_n(X_{i-}))^2} \left[\frac{1}{n} \sum_{j=1}^n \frac{\varphi_n(X_j)}{1 - G_n(X_{j-})} \delta_j 1_{\{X_j \geq X_i\}} \right]^2, \end{aligned}$$

where

$$\varphi_n(x) = 12xF_n(x) - 6x - 2 - 12 \frac{1}{n} \sum_{i=1}^n \frac{X_i \delta_i}{1 - G_n(X_{i-})} 1_{\{X_i \leq x\}} + 6 \frac{1}{n} \sum_{i=1}^n \frac{X_i \delta_i}{1 - G_n(X_{i-})}.$$

and G_n is a Kaplan-Meier estimator given by

$$1 - G_n(x) = \prod_{X_i \leq x} \left[1 - \frac{1 - \delta_i}{\sum_{k=1}^n 1_{\{X_k \geq X_i\}}} \right].$$

Before we may define the goodness-of-fit test, we need to show the consistency of the variance estimator σ_n^2 . For this, we require an assumption which is stronger than A1. In particular:

A3 : There exists $\varepsilon > 0$ such that $\int_0^1 \frac{F(dx)}{(1 - G(x))^{1+\varepsilon}} < \infty$

Lemma 1 Under A3, we have

$$\sigma_n^2 \xrightarrow{P} \sigma^2$$

Proof. See Appendix.

Finally, we have

Theorem 2 Under H_0 and assumptions A2 and A3, we have that

$$T_n := \frac{\sqrt{n}\tilde{Q}_n}{\sqrt{\sigma_n^2}} \xrightarrow{d} N(0, 1). \quad (8)$$

Proof. The result follows from Corollary 1 and Lemma 1. This completes the proof. ■

In view of Theorem 1, we reject H_0 at level α if

$$T_n \leq \Phi^{-1}(\alpha/2) \text{ or } T_n \geq \Phi^{-1}(1 - \alpha/2),$$

where Φ^{-1} is the inverse of the standard normal cdf.

Additionally, under the Koziol-Green model and in view of Corollary 2, we define

$$T_n^{KG} := \frac{\sqrt{n}\tilde{Q}_n}{\sqrt{\hat{\sigma}_{KG}^2}}, \quad (9)$$

where

$$\hat{\sigma}_{KG}^2 = \frac{-\hat{\beta}^4 + 4\hat{\beta}^3 - 17\hat{\beta}^2 + 38\hat{\beta} - 24}{(\hat{\beta} - 1)(\hat{\beta} - 2)(\hat{\beta} - 3)(\hat{\beta} - 4)(\hat{\beta} - 5)}$$

and

$$\hat{\beta} = \left(\int (1 - G_n(x))F_n(dx) \right)^{-1} - 1.$$

It is easy to see that

$$\int (1 - G_n(x))F_n(dx) \xrightarrow{P} \int (1 - G(x))F(dx).$$

Hence $\hat{\beta} \xrightarrow{P} \beta$ and $\hat{\sigma}_{KG}^2 \xrightarrow{P} \sigma^2$. Consequently, as before, we reject H_0 at level α if

$$T_n^{KG} \leq \Phi^{-1}(\alpha/2) \text{ or } T_n^{KG} \geq \Phi^{-1}(1 - \alpha/2).$$

5. Simulation study

Here we conduct an extensive simulation study to show the behaviour of our test. In the following subsection we consider only the null hypothesis, while in Subsection 5.2 we include the power study under different families of alternatives. In both subsections we compare our method with the Pearson-type goodness-of-fit test proposed by Akritas (1988). Following the notation of Section 4, we denote by T_n and T_n^{KG} our test statistics for the general censoring and under the Koziol-Green model, respectively. See, equations (8) and (9) for details. Moreover, we denote by $A_{(nc)}$ the χ^2 test proposed by Akritas (1988), where nc denotes the number of cells.

5.1. Null hypothesis

In this section we present the results of the proposed methods under the null hypothesis and at 5% significance level. As mentioned before, we consider our tests T_n and T_n^{KG} ,

together with the test presented by Akritas (1988). Following the latter work, we consider $nc = 2$ and $nc = 5$ and denote these tests by $A_{(2)}$ and $A_{(5)}$, respectively. The results are based on 5000 trials. From Table 1 we see that tests T_n and those from Akritas hold very well the significance level. The test based on the Koziol-Green model holds the 5% level when censoring is low. However, for more than 20% of missing data, the variance σ_{KG}^2 does not captures the variability of our \tilde{Q}_n correctly and, therefore, the significance level is slightly overestimated for heavy censoring.

Table 1: Empirical level for testing null hypothesis.

	10% censoring				20% censoring				30% censoring			
	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}
$n = 50$	0.0508	0.0552	0.0538	0.0578	0.0450	0.0560	0.0558	0.0530	0.0496	0.0596	0.0560	0.0648
$n = 100$	0.0480	0.0582	0.0548	0.0532	0.0478	0.0530	0.0492	0.0560	0.0502	0.0568	0.0528	0.0708
$n = 200$	0.0468	0.0524	0.0574	0.0498	0.0522	0.0508	0.0508	0.0604	0.0494	0.0522	0.0468	0.0666

5.2. Power study

In order to study the power of our test we consider two different models:

Model 1: To test the uniformity ($H_0 : F = F_U$) we choose three parametric families of alternative probability distributions with support on $[0, 1]$:

- (a) Lehmann alternatives,

$$F_\theta(x) = x^\theta, 0 \leq x \leq 1, \theta \geq 1;$$

where for $\theta = 1$ we have $F_\theta = F_U$.

- (b) compressed uniform alternatives,

$$F_\theta(x) = \frac{x - \theta}{1 - 2\theta}, \theta \leq x \leq 1 - \theta,$$

where $0 \leq \theta \leq 1/2$; and for $\theta = 0$ we have $F_\theta = F_U$.

- (c) centred distributions having a U-shaped density for $\theta \in (0, 1)$, or wedge-shaped density for $\theta > 1$

$$F_\theta(x) = \begin{cases} \frac{1}{2}(2x)^\theta, & 0 \leq x \leq 1/2 \\ 1 - \frac{1}{2}(2(1-x))^\theta, & 1/2 \leq x \leq 1 \end{cases}$$

where for $\theta = 1$ we have $F_\theta = F_U$.

Model 2: An exponentiality test (with parameter $\lambda = 1$), where the alternatives are Weibull distributions with parameters 1 and θ . More precisely, $F_\theta(x) = 1 - e^{-x^\theta}$, where $\theta = 1$ gives us the exponential distribution of the null hypothesis.

Additionally, the censoring variable C is generated under the Koziol-Green model. That is, $1 - G(x) = (1 - F(x))^\beta$, where $\beta = \frac{p}{1-p}$ and $p = \mathbb{P}(X > C)$ is the censoring level.

In the following figures and tables we present the power study at a 5% significance level. Panels (a1)-(c3) of Figure 2 contain the power of the test for Model 1 and panels (d1)-(d3) of Figure 2 contain the power under Model 2, for different sample sizes ($n = 50, 100, 200$) and one censoring level of 20%. All those figures are based on 2000 trials.

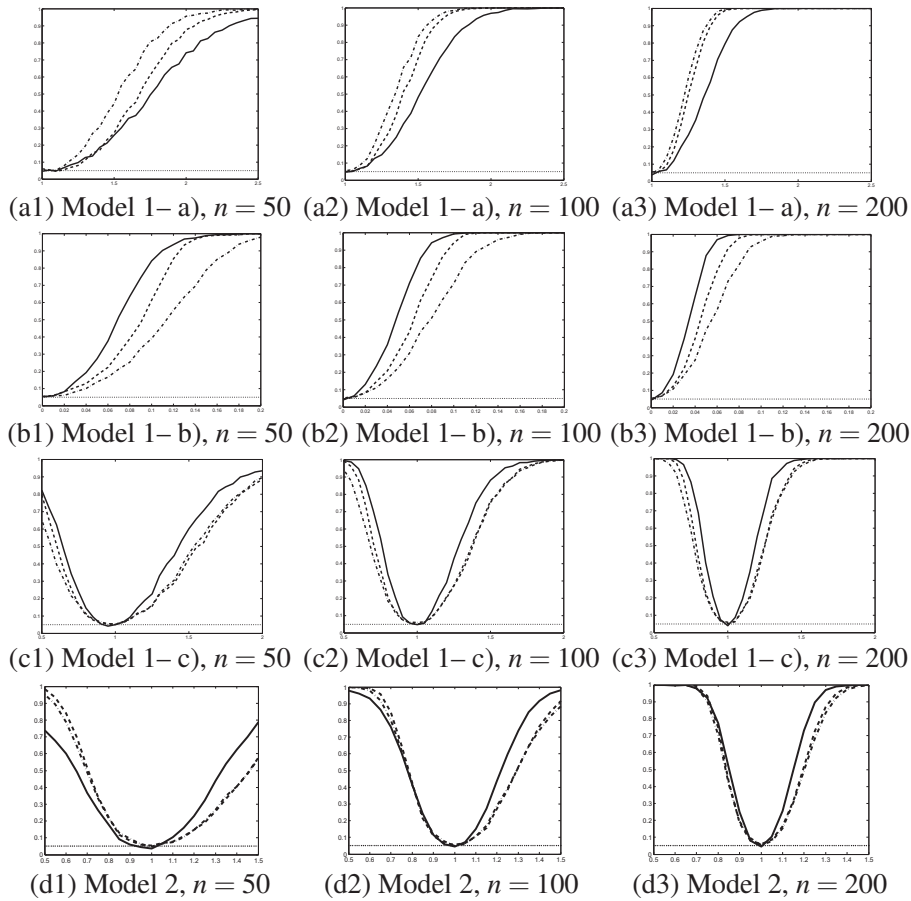


Figure 2: Power study for Model 1 (a1–c3) and Model 2 (d1–d3) for three different sample sizes and censoring rate $p = 0.2$. T_n (solid line), $A_{(5)}$ (dashed line) and $A_{(2)}$ (dash-dotted line).

Table 2: Power study for Model 1 and Model 2.

Model 1, Alternative a)												
$n = 100$	$p = 0.1$				$p = 0.2$				$p = 0.3$			
	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}
$\theta = 1.5$	0.5277	0.7107	0.8040	0.5547	0.4760	0.7220	0.8190	0.5167	0.3957	0.7093	0.8260	0.4570
$\theta = 2$	0.9847	0.9997	0.9997	0.9873	0.9997	1	1	1	0.9070	0.9980	0.9995	0.9520
$\theta = 2.5$	1	1	1	1	0.9997	1	1	1	0.9237	0.9990	0.9997	0.9607
$n = 200$	$p = 0.1$				$p = 0.2$				$p = 0.3$			
	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}
$\theta = 1.5$	0.8410	0.9823	0.9887	0.8540	0.8020	0.9830	0.9893	0.8200	0.7057	0.9833	0.9910	0.7517
$\theta = 2$	0.9997	1	1	0.9997	1	1	1	1	0.9950	1	1	0.9990
$\theta = 2.5$	1	1	1	1	1	1	1	1	0.9983	1	1	0.9997
Model 1, Alternative b)												
$n = 100$	$p = 0.1$				$p = 0.2$				$p = 0.3$			
	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}
$\theta = 0.05$	0.5987	0.3897	0.2940	0.5473	0.5370	0.3027	0.2297	0.5440	0.4563	0.2530	0.1803	0.5037
$\theta = 0.1$	1	0.9837	0.8510	0.9987	0.9957	0.9413	0.7163	0.9950	0.9410	0.9000	0.5910	0.9820
$\theta = 0.15$	1	1	0.9963	1	1	1	0.9797	1	0.9940	1	0.9500	1
$n = 200$	$p = 0.1$				$p = 0.2$				$p = 0.3$			
	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}
$\theta = 0.05$	0.9133	0.7517	0.5820	0.8830	0.8560	0.6003	0.4250	0.8313	0.7683	0.4870	0.3287	0.7973
$\theta = 0.1$	1	1	0.9933	1	1	1	0.9570	1	0.9977	0.9987	0.8977	1
$\theta = 0.15$	1	1	1	1	1	1	1	1	1	1	0.9993	1
Model 1, Alternative c)												
$n = 100$	$p = 0.1$				$p = 0.2$				$p = 0.3$			
	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}
$\theta = 0.75$	0.6587	0.3623	0.3217	0.6803	0.5337	0.3410	0.2877	0.6117	0.3490	0.3167	0.2823	0.5133
$\theta = 1.25$	0.4903	0.3090	0.3230	0.5347	0.4230	0.2667	0.2757	0.4957	0.3740	0.2323	0.2423	0.4720
$\theta = 1.5$	0.9513	0.8050	0.7987	0.9643	0.8787	0.7287	0.7353	0.9333	0.7907	0.6720	0.6800	0.9110
$n = 200$	$p = 0.1$				$p = 0.2$				$p = 0.3$			
	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}
$\theta = 0.75$	0.9337	0.7217	0.6167	0.9423	0.8933	0.6783	0.5860	0.9167	0.7307	0.6363	0.5423	0.8223
$\theta = 1.25$	0.7920	0.5513	0.5537	0.8103	0.7280	0.4823	0.4993	0.7773	0.6243	0.4277	0.4437	0.7303
$\theta = 1.5$	0.9993	0.9817	0.9750	0.9993	0.9980	0.9740	0.9623	0.9993	0.9613	0.9433	0.9233	0.9937
Model 2. Power study for $\theta = 1 + Hn^{-0.5}$												
$n = 100$	$p = 0.1$				$p = 0.2$				$p = 0.3$			
	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}
$H = -4$	0.9720	0.9960	0.9872	0.9914	0.9346	0.9928	0.9754	0.9844	0.7752	0.9822	0.9564	0.9328
$H = -2$	0.5580	0.4828	0.4746	0.6094	0.4236	0.4118	0.3966	0.5528	0.2354	0.3684	0.3510	0.4364
$H = 2$	0.4784	0.3112	0.3118	0.4954	0.4278	0.2608	0.2736	0.4730	0.3910	0.2194	0.2302	0.4578
$H = 4$	0.9594	0.8464	0.8296	0.9598	0.9160	0.7564	0.7430	0.9412	0.8204	0.6572	0.6478	0.9056
$n = 200$	$p = 0.1$				$p = 0.2$				$p = 0.3$			
	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}	T_n	$A_{(5)}$	$A_{(2)}$	T_n^{KG}
$H = -4$	0.9876	0.9938	0.9856	0.9940	0.9704	0.9820	0.9604	0.9864	0.8664	0.9682	0.9306	0.9490
$H = -2$	0.5712	0.4390	0.4406	0.5976	0.4636	0.3870	0.3808	0.5312	0.2892	0.3370	0.3338	0.4268
$H = 2$	0.5108	0.3312	0.3354	0.5284	0.4562	0.2768	0.2818	0.4962	0.3998	0.2302	0.2410	0.4736
$H = 4$	0.9686	0.8946	0.8610	0.9676	0.9446	0.8216	0.7830	0.9550	0.8764	0.7330	0.7032	0.9316

Moreover, the results on Table 2 are based on 5000 trials and show the power under alternatives for two different sample sizes $n = 100, 200$, censoring levels $p = 0.1, 0.2, 0.3$ and different values of parameter θ . In particular, for Model 2, we choose $\theta = 1 + Hn^{-0.5}$ and $H \in \{-4, -2, 2, 4\}$. Both, tables and figures, include a comparison to the Pearson-type test proposed by Akritas (1988). As before, we use the number of cells (nc) equal to 2 and 5.

The goal here is to show the changes in power when varying both θ parameter and the censoring rate p . In particular, Figure 2 is devoted to illustrate the changes in power when considering a given range for $\theta \in \Theta$. That is, Figure 2 contains the power curves of the statistic for all the alternatives, for $n = 50$ and a fixed moderate censoring rate of $p = 0.2$. On the other hand, Table 2 is devoted to show the changes in power when considering different censoring rates. Therefore, Table 2 contains the power study for the remaining sample sizes, $n = 100, 200$, for three fixed values of θ and different censoring rates $p = 0.1, 0.2, 0.3$.

Concerning the uniformity test (Model 1), it is clear that for alternatives (b) and (c) our test outperforms that proposed by Akritas. Additionally, our test neither depends on the number of cells nor on the choice of cell boundaries. The influence of the number of cells in Akritas proposal is made obvious in panels (a1)–(c3) of Figure 2. While $A_{(2)}$ gives better results than $A_{(5)}$ for alternative (a), the opposite can be observed for alternatives (b) and (c). Unfortunately, the modification of the maximum correlation coefficient exhibits also some weak points. That is, the alternative (a) for $\theta \in (0, 1)$ does not provide satisfactory results, since $Q = 1$ for $\theta = 0.5$. Regarding the exponentiality test (Model 2), we get better results than the competitive test of Akritas (1988) when the alternative is Weibull with parameter $\theta > 1$. For $\theta < 1$, our test reaches the high power of the Pearson-type test for big sample sizes. However, notice that in Model 2 and for all the considered values of θ , the test statistic under the Koziol-Green model, T_n^{KG} , gives very good results independently on the sample size.

6. Further extensions and application

6.1. Composite null hypothesis

So far, our test T_n has been designed to test a fully specified null hypothesis. It does strongly depend on the fact that the transformed lifetime $F_0(X)$ is $[0, 1]$ uniformly distributed under $H_0 : F = F_0$. In this section we consider a more general case, that is, when the distribution function to be tested depends on an unknown parameter λ . Let now consider the following null hypothesis

$$H_0 : F \in \{F_\lambda : \lambda \in \mathbb{R}^d\}.$$

In this case, first we need to estimate the parameter λ using, e.g., a maximum-likelihood estimator $\hat{\lambda}$. Clearly, if F_λ is twice differentiable in λ and the estimator $\hat{\lambda}$ is \sqrt{n} consistent, by the Taylor expansion we have that $F_{\hat{\lambda}}(X) = U + O_P(n^{-1/2})$, where $U = F_\lambda(X) \sim U[0, 1]$ under the null hypothesis H_0 . The test statistic \tilde{Q}_n should still admit a normal limit but the error term enters the variance of our test statistic and hence the asymptotic variance given in Theorem 1 is no longer valid. Even though the theoretical properties of our test in the case of such a composite hypothesis are beyond the scope of this paper, to test this kind of hypothesis we propose a modified test with a jackknife estimator of the variance, which does take into account the estimation of the parameters and works very well in practice. Preliminary simulation studies, as those given in Figure 3, confirm the normality of the statistic and adequacy of the variance. We proceed as follows:

1. Based on the sample X_1, \dots, X_n , find the maximum-likelihood estimator (MLE) $\hat{\lambda}$.
2. Define the pseudo-values $\tilde{X}_i = F_{\hat{\lambda}}(X_i)$ for $i = 1, \dots, n$.
3. Based on the sample $\tilde{X}_1, \dots, \tilde{X}_n$, compute the test statistic \tilde{Q}_n defined in (7).
4. Compute the jackknife estimator of the variance following the steps:
 - For every $i = 1, \dots, n$, choose the subsample $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n$ and compute the MLE $\hat{\lambda}^{(-i)}$.
 - Define the pseudo-values $\tilde{X}_j = F_{\hat{\lambda}^{(-i)}}(X_j)$ for $j = 1, \dots, i-1, i+1, \dots, n$.
 - Based on the sample $\tilde{X}_1, \dots, \tilde{X}_{i-1}, \tilde{X}_{i+1}, \dots, \tilde{X}_n$, compute the test statistic $\tilde{Q}_n^{(-i)}$.
 - Set

$$nV_n(\tilde{Q}_n) = (n-1) \sum_{i=1}^n (\tilde{Q}_n^{(-i)} - \bar{Q}_n)^2,$$

$$\text{where } \bar{Q}_n = \frac{1}{n} \sum_{i=1}^n \tilde{Q}_n^{(-i)}.$$

5. Define the test statistic

$$J_n := \frac{\sqrt{n}\tilde{Q}_n}{\sqrt{nV_n(\tilde{Q}_n)}}.$$

6. Reject H_0 if

$$J_n \leq \Phi^{-1}(\alpha/2) \text{ or } J_n \geq \Phi^{-1}(1 - \alpha/2).$$

In order to check the behaviour of this new jackknife-test J_n , we study the hypothesis $H_0 : F \in \{\exp(\lambda) : \lambda \in (0, \infty)\}$, where the alternatives come from the Weibull distribution. Our simulated sample comes from $\exp(\lambda = 1)$ and λ is estimated using maximum likelihood. In Figure 3 we compare the test based in T_n , defined in equation (8)

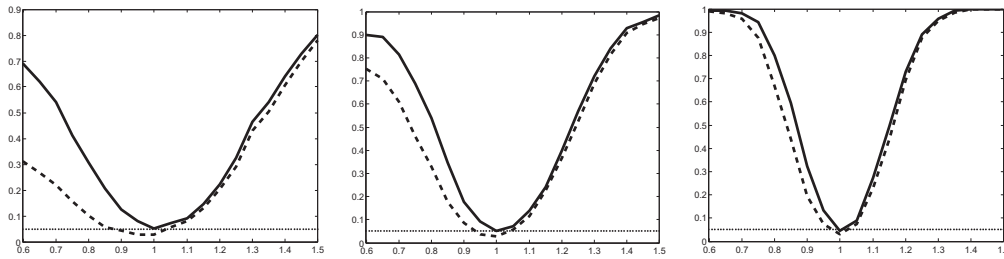


Figure 3: Power study: J_n (solid line) and T_n (dashed line), where $n = 50$ (left), $n = 100$ (middle) and $n = 200$ (right), censoring rate $p = 0.2$

of Section 4, with that based on J_n . As expected, the new test based on J_n gives very good results: The variance estimator adapted to the composite hypothesis is performing very well, leading to a more powerful test. The differences in power between both statistics seem to decrease with the sample size. Nevertheless, the theoretical properties of J_n are out of scope of the present paper.

6.2. Real data example

We illustrate the use of our test on the head-and-neck cancer data from Nikulin and Haghighi (2006). These authors fitted the Generalized-Power Weibull distribution $F(x, \sigma, \nu, \gamma)$ to the data. Motivated by the boxplot in Figure 4, we remove several observations which could be considered as outliers. This gives us 44 observations with around 11% censoring rate. We perform a goodness-of-fit test for the before-mentioned Generalized-Power Weibull distribution $F_0^a(x, \sigma, \nu, \gamma) = F(x, \sigma, \nu, \gamma)$. Additionally, we also consider the Weibull distribution $F_0^b(x, \sigma, \nu) = F(x, \sigma, \nu, 1)$ and the Exponential distribution $F_0^c(x, \sigma) = F(x, \sigma, 1, 1)$, where

$$F(x, \sigma, \nu, \gamma) = 1 - \exp\left(1 - (1 + (x/\sigma)^\nu)^{1/\gamma}\right).$$

First, we fitted the parameters using MLE under random censoring obtaining the estimators $(\hat{\sigma}, \hat{\nu}, \hat{\gamma})$ and the following distributions $F_0^a(x, 4.63, 1.82, 1.91)$, $F_0^b(x, 1.44, 8.45)$ and $F_0^c(x, 8.33)$. Then we applied our test J_n and obtained the following p-values: $p^a = 0.86$, $p^b = 0.88$ and $p^c = 0.01$ for the Generalized-Power Weibull, Weibull and Exponential, respectively. Hence, the results of the test confirm what Figure 4 shows, that both Generalized-Power Weibull and Weibull fit the data very well, whereas the Exponential distribution is not adequate to describe the head-and-neck cancer data.

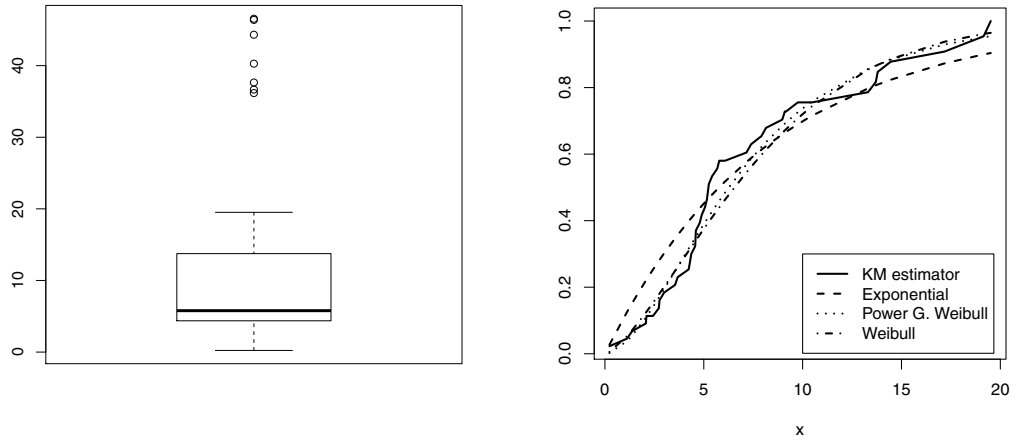


Figure 4: Boxplot (left) and Kaplan-Meier estimator (right) together with $F_0^a(x, \hat{\sigma}, \hat{\nu}, \hat{\gamma})$ (dashed), $F_0^b(x, \hat{\sigma}, \hat{\nu})$ (dotted) and $F_0^c(x, \hat{\sigma})$ (dot-dashed) for the head-and-neck cancer data.

7. Conclusions

In this work we developed and studied a goodness-of-fit test based on maximum correlation under random censoring. The advantage of our test over other goodness-of-fit competitors, like χ^2 test proposed by Akritas (1988), is its simplicity. Our test is asymptotically normally distributed and neither the number of classes nor the class boundaries have to be chosen. The proposed test outperforms that by Akritas (1988) for most of the alternatives studied. Even though the test was designed to check uniformity, with a simple transformation it can be applied to any, fully specified, continuous distribution. Finally, it can be extended to composite hypothesis, that is, when the distribution in the null hypothesis is known up to a parameter. A jackknife modification for the asymptotic variance has been proposed. A theoretical study of the test under the composite null hypothesis is out of the scope of the present paper and purpose of further research.

8. Appendix

Proof of Theorem 1

In view of (7), we can write \tilde{Q}_n in the following way

$$\tilde{Q}_n = \int_0^1 \int_0^1 \tilde{h}(x, y) F_n(dx) F_n(dy), \quad (10)$$

where

$$\tilde{h}(x, y) = (6x - 2)1_{\{y < x\}} - 6x1_{\{y > x\}}.$$

In the first step of the proof we write \tilde{Q}_n as a sum of four terms as follows

$$\tilde{Q}_n = \tilde{Q}_1 + \tilde{Q}_{2n} + \tilde{Q}_{3n} + \tilde{Q}_{4n},$$

where

$$\begin{aligned}\tilde{Q}_1 &= \int_0^1 \int_0^1 \tilde{h}(x, y) F(dx) F(dy) \\ \tilde{Q}_{2n} &= \int_0^1 \int_0^1 \tilde{h}(x, y) (F_n(dx) - F(dx)) F(dy) \\ \tilde{Q}_{3n} &= \int_0^1 \int_0^1 \tilde{h}(x, y) (F_n(dy) - F(dy)) F(dx) \\ \tilde{Q}_{4n} &= \int_0^1 \int_0^1 \tilde{h}(x, y) (F_n(dx) - F(dx)) (F_n(dy) - F(dy)).\end{aligned}$$

By (6) and since F is continuous, we have that $\tilde{Q}_1 = Q^1$. As to $\tilde{Q}_{2n} + \tilde{Q}_{3n}$, we obtain

$$\tilde{Q}_{2n} + \tilde{Q}_{3n} = \int_0^1 \varphi(x) (F_n(dx) - F(dx)),$$

where

$$\begin{aligned}\varphi(x) &= \int_0^1 \tilde{h}(y, x) F(dy) + \int_0^1 \tilde{h}(x, y) F(dy) \\ &= 12xF(x) - 6x - 2 - 12 \int_0^x yF(dy) + 6 \int_0^1 yF(dy).\end{aligned}$$

It remains to show that $\tilde{Q}_{4n} = o_{\mathbb{P}}(n^{1/2})$. For this, set $\tau_{\tilde{H}} = \inf\{t : \tilde{H}(t) = 1\}$, where $\tilde{H}(t) = \mathbb{P}(X \leq t)$ is the cdf of the observed sample. Since the support $\text{supp}(F) \in [0, 1]$ and G fulfills assumption A1, we have that $\tau_{\tilde{H}} = 1$. Moreover, by definition of $\tilde{h}(x, y)$, we can show that

$$\tilde{Q}_{4n} = -12 \int_0^1 x(F_n(x) - F(x))(F_n(dx) - F(dx)) - 2(F_n(1) - F(1))^2 =: \tilde{Q}_{4n}^a + \tilde{Q}_{4n}^b.$$

Now, we may consider the two terms, \tilde{Q}_{4n}^a and \tilde{Q}_{4n}^b , separately. According to Theorem 2 (7) in Ying (1989) and under A1, the process $\sqrt{n}(F_n - F)$ converges weakly to a Brownian process. See, also equation (11) in Wellner (2007). More precisely,

$$\sqrt{n}(F_n - F) \rightarrow (1 - F)\mathbb{B}(C), \text{ in } D[0, \tau_{\bar{H}}],$$

where $\mathbb{B}(C)$ is a Brownian process and $D[0, \tau_{\bar{H}}]$ denotes the Skorohod space. Furthermore, since F is continuous and D^0 is a set of uniformly bounded functions, we have that $\sqrt{n}(F_n - F) \in D^0$ with probability exceeding $1 - \varepsilon$ for every $\varepsilon > 0$. Additionally, $x \in [0, 1]$ and $\sup_{x \in [0, \tau_{\bar{H}}]} |F_n(x) - F(x)| \rightarrow 0$ almost surely. Hence, using Theorem 2.1 in Rao (1962) with $g(x) = \sqrt{n}(F_n(x) - F(x))x$, we obtain

$$\sqrt{n}\tilde{Q}_{4n}^a = -12 \int_0^1 g(x)(F_n(dx) - F(dx)) = o_{\mathbb{P}}(1).$$

Additionally, under A1, $F_n(1) - F(1) = O_{\mathbb{P}}(\sqrt{n})$ and hence $\sqrt{n}\tilde{Q}_{4n}^b = o_{\mathbb{P}}(1)$. Notice that, $F_n(1) - F(1) = \int_0^1 1(F_n(dx) - F(dx))$. Hence we apply the results from Stute (1995) for $\varphi(x) = 1$.

Finally, we obtain the following representation

$$\tilde{Q}_n = Q^1 + \int_0^1 \varphi(x)(F_n(dx) - F(dx)) + o_{\mathbb{P}}(n^{1/2}).$$

The asymptotic normality is now a direct consequence of Stute (1995). More precisely, under A1 and A2, we obtain

$$\sqrt{n} \int_0^1 \varphi(x)(F_n(dx) - F(dx)) \rightarrow \mathcal{N}(0, \sigma^2).$$

This completes the proof. ■

Proof of Lemma 1

Recall, that

$$\begin{aligned} \sigma_n^2 &= \int_0^1 \frac{\varphi_n^2(x)}{1 - G_n(x-)} F_n(dx) - \left[\int_0^1 \varphi_n(x) F_n(dx) \right]^2 - \\ &\quad - \int_0^1 \left[\int_x^1 \varphi_n(y) F_n(dy) \right]^2 \frac{(1 - F_n(x))G_n(dx)}{(1 - H_n(x))^2} \\ &=: A_{1n} - A_{2n} - A_{3n}. \end{aligned}$$

where

$$\varphi_n(x) = 12xF_n(x) - 6x - 2 - 12 \int_0^x yF_n(dy) + 6 \int_0^1 yF_n(dy).$$

By consistency of the Kaplan-Meier estimator, we have $\varphi_n(x) \rightarrow \varphi(x)$ in probability. Let us consider the first term in the representation of σ_n^2 . Since $|\varphi_n(x)| \leq K_1 = \text{constant}$, $G_n(x) \rightarrow G(x)$ in probability,

$$\max_{i=1, \dots, n} \frac{1 - G(X_{i-})}{1 - G_n(X_{i-})} = O_P(1)$$

by Zhou (1991) and

$$\frac{1}{1 - G_n(x-)} = \frac{1}{1 - G(x-)} + \frac{G_n(x-) - G(x-)}{(1 - G_n(x-))(1 - G(x-))}$$

we have

$$A_{1n} = \int_0^1 \frac{\varphi^2(x)}{1 - G(x)} F_n(dx) + o_P(1)$$

Finally, by Theorem 1.1. in Stute and Wang (1993), $\int \Phi(x) F_n(dx) \rightarrow \int \Phi(x) F(dx)$ with probability 1 and hence in probability, provided that $\int |\Phi(x)| F(dx) < \infty$. Hence, by A3, we obtain

$$A_{1n} \xrightarrow{P} \int_0^1 \frac{\varphi^2(x)}{1 - G(x)} F(dx).$$

Obviously, we have

$$A_{2n} \xrightarrow{P} \left[\int_0^1 \varphi(x) F(dx) \right]^2.$$

Finally, similarly as we have done for A_{1n} , we may show that

$$A_{3n} = \int_0^1 \left[\int_x^1 \varphi(y) F(dy) \right]^2 \frac{(1 - F(x)) G_n(dx)}{(1 - H(x))^2} + o_P(1)$$

By A3 and since $|\varphi(y)| \leq K_2 = \text{constant}$ we obtain

$$\begin{aligned} \int_0^1 \left[\int_x^1 \varphi(y) F(dy) \right]^2 \frac{(1 - F(x)) G(dx)}{(1 - H(x))^2} &\leq K_2^2 \\ \int_0^1 \left[\int_x^1 \frac{F(dy)}{(1 - G(y))^{1+\varepsilon}} \right] \frac{G(dx)}{(1 - G(x))^{1-\varepsilon}} &< \infty. \end{aligned}$$

Hence, by Theorem 1.1. in Stute and Wang (1993),

$$A_{3n} \rightarrow \int_0^1 \left[\int_x^1 \varphi(y) F(dy) \right]^2 \frac{(1 - F(x)) G(dx)}{(1 - H(x))^2}$$

in probability. This completes the proof. ■

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