

**Supplemental material for “Unusual-event processes  
for count data”**

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June 2022

The material contained herein is supplementary to the article named  
in the title and published in SORT-Statistics and Operations  
Research Transactions Volume 46(1).

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## 1. Proof that the CMP model is not a renewal process

Let  $\{Y_k, k \in \mathbb{N}\}$  denote a sequence of interarrival times between the  $(k-1)$ th and  $k$ th event, and let  $X(t)$  be a discrete random variable, representing the total number of events that occur before or at exactly time  $t$ . The arrival time  $S_n$  is the time of the  $n$ th event. It can be computed by the sum of the interarrival times,  $S_n = \sum_{k=1}^n Y_k$ . The probability function of the count variable  $X(t)$  is given by

$$P_n(t) = \begin{cases} 1 - F_{S_1}(t) & \text{for } n = 0 \\ \int_0^t f_{S_n}(s_n) [1 - F_{Y_{n+1}}(t - s_n)] ds_n & \text{for } n = 1, 2, \dots, \end{cases} \quad (1)$$

where  $f_{S_n}(t)$  represents the probability density function of  $S_n$ , and  $F_{Y_n}(t)$  represents the cumulative distribution function of  $Y_n$ . The CMP probability distribution function is

$$P_n(t) = \frac{(\lambda t)^n}{(n!)^\nu Z(\lambda t, \nu)}, \quad (2)$$

where  $Z(\lambda t, \nu) = \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{(j!)^\nu}$ . If we let  $\nu = 1$ , then  $P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$  (Poisson distribution). We prove that the CMP model is not a renewal process by showing that  $F_{Y_1}(t) \neq F_{Y_2}(t)$ .

### Derivation of $F_{Y_1}(t)$

Putting  $n = 0$  in Equation (2), we obtain

$$P_0(t) = \frac{1}{Z(\lambda t, \nu)}, \quad \text{and} \quad F_{Y_1}(t) = F_{S_1}(t) = 1 - \frac{1}{Z(\lambda t, \nu)}.$$

Letting  $\phi_1(t) = 1 - F_{Y_1}(t) = \sum_{i=0}^{\infty} c_i (\lambda t)^i$ , the following equation is obtained:

$$\begin{aligned} \phi_1(t) Z(\lambda t, \nu) &= 1 \\ \sum_{i=0}^{\infty} c_i (\lambda t)^i \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{(j!)^\nu} &= 1 \\ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{c_i}{(j!)^\nu} (\lambda t)^{i+j} &= 1 \\ \sum_{k=0}^{\infty} \left( \sum_{i=0}^k \frac{c_i}{((k-i)!)^\nu} \right) (\lambda t)^k &= 1 \end{aligned}$$

This equation can be solved recursively in the usual manner:

$$\begin{aligned}
c_0 &= 1 \\
c_1 &= -\frac{c_0}{(1!)^v} = -1 \\
c_2 &= -\left(\frac{c_0}{(2!)^v} + \frac{c_1}{(1!)^v}\right) = -\frac{1}{(2!)^v} + 1 \\
c_3 &= -\left(\frac{c_0}{(3!)^v} + \frac{c_1}{(2!)^v} + \frac{c_2}{(1!)^v}\right) = -\frac{1}{(3!)^v} + \frac{2}{(2!)^v} - 1
\end{aligned}$$

From these equations for  $c_0$ ,  $c_1$ ,  $c_2$ , and  $c_3$ , one can deduce that the general solution might be of the form

$$c_i = \begin{cases} 1 & \text{for } i = 0 \\ -\sum_{j=0}^{i-1} \frac{c_j}{((i-j)!)^v} & \text{for } i > 0. \end{cases}$$

Therefore,

$$F_{Y_1}(t) = \lambda t + \left(\frac{2}{(2!)^v} - 2\right) \frac{(\lambda t)^2}{2!} + \left(\frac{6}{(3!)^v} - \frac{12}{(2!)^v} + 6\right) \frac{(\lambda t)^3}{3!} + \dots$$

If we let  $v = 1$ , then  $F_{Y_1}(t) = 1 - e^{-\lambda t}$  (Poisson process).

### Derivation of $F_{Y_2}(t)$

Putting  $n = 1$  in Equation (2), we obtain

$$P_1(t) = \frac{\lambda t}{Z(\lambda t, v)} = \sum_{i=0}^{\infty} c_i (\lambda t)^{i+1} = \int_0^t f_{S_1}(s_1) \phi_2(t - s_1) ds_1,$$

where  $\phi_2(t) = 1 - F_{Y_2}(t) = \sum_{i=0}^{\infty} \phi_2^{(i)}(0) \frac{t^i}{i!}$ . Using the Leibniz integral rule, the first few derivatives of  $P_1(t)$  are as follows:

$$\begin{aligned}
\sum_{i=0}^{\infty} c_i \lambda (i+1) (\lambda t)^i &= f_{S_1}(t) \phi_2(0) + \int_0^t f_{S_1}(s_1) \phi_2'(t - s_1) ds_1 \\
\sum_{i=1}^{\infty} c_i \lambda^2 (i+1)(i) (\lambda t)^{i-1} &= f_{S_1}'(t) \phi_2(0) + f_{S_1}(t) \phi_2'(0) + \int_0^t f_{S_1}(s_1) \phi_2''(t - s_1) ds_1 \\
\sum_{i=2}^{\infty} c_i \lambda^3 \left(\prod_{k=-1}^1 (i+k)\right) (\lambda t)^{i-2} &= \sum_{j=0}^2 f_{S_1}^{(2-j)}(t) \phi_2^{(j)}(0) + \int_0^t f_{S_1}(s_1) \phi_2'''(t - s_1) ds_1 \\
\sum_{i=3}^{\infty} c_i \lambda^4 \left(\prod_{k=-2}^1 (i+k)\right) (\lambda t)^{i-3} &= \sum_{j=0}^3 f_{S_1}^{(3-j)}(t) \phi_2^{(j)}(0) + \int_0^t f_{S_1}(s_1) \phi_2^{(4)}(t - s_1) ds_1
\end{aligned}$$

Taking  $t = 0$ , we have

$$c_0 \lambda = f_{S_1}(0) \phi_2(0) \quad (3)$$

$$2!c_1 \lambda^2 = f'_{S_1}(0) \phi_2(0) + f_{S_1}(0) \phi'_2(0) \quad (4)$$

$$3!c_2 \lambda^3 = f''_{S_1}(0) \phi_2(0) + f'_{S_1}(0) \phi'_2(0) + f_{S_1}(0) \phi''_2(0) \quad (5)$$

$$4!c_3 \lambda^4 = f'''_{S_1}(0) \phi_2(0) + f''_{S_1}(0) \phi'_2(0) + f'_{S_1}(0) \phi''_2(0) + f_{S_1}(0) \phi'''_2(0) \quad (6)$$

The first few derivatives of  $Z(\lambda t, \nu) = \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{(j!)^\nu}$  are as follows:

$$Z'(\lambda t, \nu) = \sum_{j=1}^{\infty} j \lambda \frac{(\lambda t)^{j-1}}{(j!)^\nu}$$

$$Z''(\lambda t, \nu) = \sum_{j=2}^{\infty} j(j-1) \lambda^2 \frac{(\lambda t)^{j-2}}{(j!)^\nu}$$

$$Z'''(\lambda t, \nu) = \sum_{j=3}^{\infty} j(j-1)(j-2) \lambda^3 \frac{(\lambda t)^{j-3}}{(j!)^\nu}$$

$$Z^{(4)}(\lambda t, \nu) = \sum_{j=4}^{\infty} j(j-1)(j-2)(j-3) \lambda^4 \frac{(\lambda t)^{j-4}}{(j!)^\nu}$$

The first few derivatives of  $f_{S_1}(t) = F'_{Y_1}(t) = \frac{Z'(\lambda t, \nu)}{Z^2(\lambda t, \nu)}$  are as follows:

$$f'_{S_1}(t) = -2 \frac{(Z'(\lambda t, \nu))^2}{Z^3(\lambda t, \nu)} + \frac{Z''(\lambda t, \nu)}{Z^2(\lambda t, \nu)}$$

$$f''_{S_1}(t) = 6 \frac{(Z'(\lambda t, \nu))^3}{Z^4(\lambda t, \nu)} - 6 \frac{Z'(\lambda t, \nu) Z''(\lambda t, \nu)}{Z^3(\lambda t, \nu)} + \frac{Z'''(\lambda t, \nu)}{Z^2(\lambda t, \nu)}$$

$$f'''_{S_1}(t) = -24 \frac{(Z'(\lambda t, \nu))^4}{Z^5(\lambda t, \nu)} + 36 \frac{(Z'(\lambda t, \nu))^2 Z''(\lambda t, \nu)}{Z^4(\lambda t, \nu)} - 8 \frac{Z'(\lambda t, \nu) Z'''(\lambda t, \nu)}{Z^3(\lambda t, \nu)} - 6 \frac{(Z''(\lambda t, \nu))^2}{Z^3(\lambda t, \nu)} + \frac{Z^{(4)}(\lambda t, \nu)}{Z^2(\lambda t, \nu)}$$

Taking  $t = 0$ , we have  $Z(0, \nu) = 1$ ,  $Z'(0, \nu) = \lambda$ ,  $Z''(0, \nu) = \frac{2}{(2!)^\nu} \lambda^2$ ,  $Z'''(0, \nu) = \frac{6}{(3!)^\nu} \lambda^3$ , and  $Z^{(4)}(0, \nu) = \frac{24}{(4!)^\nu} \lambda^4$ . Thus,

$$f_{S_1}(0) = \lambda$$

$$f'_{S_1}(0) = \left( -2 + \frac{2}{(2!)^\nu} \right) \lambda^2$$

$$f''_{S_1}(0) = \left( 6 - \frac{12}{(2!)^\nu} + \frac{6}{(3!)^\nu} \right) \lambda^3$$

$$f'''_{S_1}(0) = \left( -24 + \frac{72}{(2!)^\nu} - \frac{48}{(3!)^\nu} - \frac{24}{(2!)^{2\nu}} + \frac{24}{(4!)^\nu} \right) \lambda^4$$

We substitute these expressions,  $c_0 = 1$ ,  $c_1 = -1$ ,  $c_2 = -\frac{1}{(2!)^v} + 1$ , and  $c_3 = -\frac{1}{(3!)^v} + \frac{2}{(2!)^v} - 1$  into Equations (3)-(6) and determine the coefficients  $\phi_2^{(n)}(0)$ :

$$\phi_2(0) = 1$$

$$\phi_2'(0) = -\frac{2}{(2!)^v} \lambda$$

$$\phi_2''(0) = \left( \frac{2}{(2!)^v} - \frac{6}{(3!)^v} + \frac{4}{(2!)^{2v}} \right) \lambda^2$$

$$\phi_2'''(0) = \left( -\frac{8}{(2!)^v} + \frac{12}{(3!)^v} - \frac{24}{(4!)^v} + \frac{4}{(2!)^{2v}} + \frac{24}{(2!)^v(3!)^v} - \frac{8}{(2!)^{3v}} \right) \lambda^3$$

Therefore,

$$\begin{aligned} F_{Y_2}(t) &= \frac{2}{(2!)^v} \lambda t + \left( -\frac{2}{(2!)^v} + \frac{6}{(3!)^v} - \frac{4}{(2!)^{2v}} \right) \frac{(\lambda t)^2}{2!} \\ &\quad + \left( \frac{8}{(2!)^v} - \frac{12}{(3!)^v} + \frac{24}{(4!)^v} - \frac{4}{(2!)^{2v}} - \frac{24}{(2!)^v(3!)^v} + \frac{8}{(2!)^{3v}} \right) \frac{(\lambda t)^3}{3!} + \dots \end{aligned}$$

If we let  $v = 1$ , then  $F_{Y_2}(t) = 1 - e^{-\lambda t}$  (Poisson process). This completes the proof.