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Alternate-wrapped circular distributions

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Abstract

To generate a circular distribution, we use the alternate-wrapping technique (unlike the usual wrapping), by wrapping in the alternate directions, after each single-wrapping. The resulting distribution is called alternate-wrapped distribution. Some general properties and distinctions between the two wrapping schemes are indicated. As an illustration, alternate-wrapped-exponential distribution and alternate-wrapped-normal distribution are considered. The moment and maximum likelihood estimator of the parameters of alternative-wrapped-exponential distribution are obtained and their performance is evaluated using simulation. Maximum likelihood estimators are obtained for the parameters of the alternate-wrapped-normal distribution and simulation study is conducted, and this distribution is used to analyse a real-life data set and is compared with the wrapped normal distribution.

MSC: *62H11, 62P12.*

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1. Introduction

In many real-life situations, characteristics of interest are not linear. For instance, wind directions, the direction of migration of birds, time of occurrence of an event in a day etcetera. These cannot be measured on a linear scale and are circular. If X is a univariate real-valued random variable then $\theta = X(\text{mod}2\pi)$ is called a wrapped circular random variable. The density function of θ is $g_w(\theta) = \sum_{m=-\infty}^{\infty} f(2m\pi + \theta), \quad 0 \leqslant \theta < 2\pi$. For illustrative examples of circular random variables and models, one may refer to Mardia and Jupp (2000) and Jammalamadaka and SenGupta (2001). Here, the density function

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of θ is obtained by wrapping the density of X around a unit radius circular cylinder in the anti-clockwise direction, so that $x = 0$ matches with $\theta = 0$. In the literature, many wrapped distributions have been developed: to mention a few, Jammalamadaka and Kozubowski (2004) have discussed some properties of wrapped exponential (WE) and wrapped Laplace distributions. Sarma, Rao, and Girija (2011) discussed the characteristic function of wrapped log-normal and wrapped Weibull distributions. Roy and Adnan (2012) have proposed a wrapped weighted exponential distribution. Adnan and Roy (2014) have introduced wrapped variance gamma distribution. Joshi and Jose (2018) have discussed wrapped Lindley distribution. Yilmaz and Bicer (2018) have introduced a new version of wrapped exponential distribution namely transmuted exponential distribution and have discussed its properties.

In this paper, we propose a new wrapping technique called "alternate-wrapping" and the generated distribution is called an alternate-wrapped distribution. To begin with, consider a density function $f(x)$ corresponding to a non-negative random variable X. Start wrapping the density around the unit radius circular cylinder in the anti-clockwise direction, by setting 0 of X with angle 0. After the first single wrapping in the anti-clockwise direction, the next single wrapping is done in the other direction, and so on. However, for the usual wrapped densities, the wrapping is continued in the same direction. The resulting alternate-wrapped density function has a period 2π. To define the density function on the entire real line, we extend the density function with periodicity 2π. In usual wrapping, for $0 < \delta < 2\pi$, the value of the density f at $\delta,\ 2\pi+\delta,\ 4\pi+\delta,\ldots$ and so on contribute to the wrapped density at $\theta = \delta$. But in alternate-wrapping, the values of the density f at $\delta, 4\pi-\delta, 4\pi+\delta, 8\pi-\delta,\ldots$ and so on contribute to the alternate wrapped density at δ. If f is an arbitrary density function then the density on the support $[0,\infty)$ is wrapped as described above and alternate wrapping of the density on the negative part starts from 0 with the first wrapping of the density (towards $-\infty$) in the clockwise direction and the next wrapping in the anti-clockwise direction and so on.

The rest of the paper is organized as follows. In Section 2, we define alternate-wrapping technique in detail and give some of its properties. In Section 3, alternate-wrapped-exponential (AWE) distribution is considered and some of its properties are given. In Section 4, the alternate-wrapped-normal (AWN) distribution is discussed. In Section 5, a data set of 506 cases of onset of lymphatic leukemia, reported in different months in the UK, is analysed using AWN and wrapped-normal (WN) distributions. Lastly, Section 6 concludes the findings.

2. Alternate-wrapping technique

Let X have a continuous distribution with density function $f(x)$. The total contribution at θ $(0 \le \theta < 2\pi)$, by the density on the domain $\{x \ge 0\}$ is

$$f^{+}(\theta) = f(\theta) + f(4\pi-\theta) + f(4\pi+\theta) + f(8\pi-\theta) + f(8\pi+\theta) + \ldots.$$

The total contribution at θ by the density on the domain $\{x<0\}$ is

$$f^-(\theta) = f(-2\pi+\theta) + f(-2\pi-\theta) + f(-6\pi+\theta) + f(-6\pi-\theta) + \dots.$$

Thus, under alternate-wrapping the total contribution at θ by the wrapping density is $f^+(\theta) + f^-(\theta)$.

Definition A circular random variable θ is said to have an *alternate-wrapped* distribution, if the density function is given by

$$g_{aw}(\theta) = f(\theta) + \sum_{m=1}^{\infty} \left(f((-1)^m 2m\pi + \theta) + f((-1)^m 2m\pi - \theta) \right), \theta \in [0, 2\pi). \tag{1}$$

For a density with positive support, the alternate-wrapped density is given by

$$g_{aw}(\theta) = f(\theta) + \sum_{m=1}^{\infty} \left(f(4m\pi - \theta) + f(4m\pi + \theta) \right), \quad \theta \in [0, 2\pi). \tag{2}$$

The alternate-wrapping of f for $x>0$ can be viewed as the usual wrapping of its modified version h (say). Under alternate-wrapping of f, for $x>0$, anti-clockwise wrapping is over intervals $(4m\pi, (4m+2)\pi)$ for $m = 0, 1, 2, \dots$, and clockwise over the other intervals. To make it anti-clockwise over $(0, \infty)$, we modify f over the intervals $((2m+1)2\pi, (2m+2)2\pi)$, to be $h(x) = f((8m+6)\pi - x)$.
Hence for $x>0$ and $m = 0, 1, 2, \dots$, let

$$h(x) = \begin{cases} f(x), & 4m\pi < x \le (4m+2)\pi \\ f((8m+6)\pi - x), & (4m+2)\pi < x \le (4(m+1))\pi. \end{cases} \tag{3}$$

Similarly, for $x \le 0$, to have clockwise wrapping through out, we need to have modification on the intervals $-(4m+4)\pi < x \le -(4m+2))\pi$. The resulting function for $x \le 0$ and $m = 0, 1, 2, \dots$ is given by

$$h(x) = \begin{cases} f(x), & -(4m+2)\pi < x \le -4m\pi \\ f(-(8m+6)\pi - x), & -(4m+4)\pi < x \le -(4m+2))\pi. \end{cases} \tag{4}$$

Hence, we have the following.

Property 2.1. *If $g_w^{(f)}(\theta), g_{aw}^{(f)}(\theta)$ are respectively wrapped and alternate-wrapped density functions generated from f, then*

$$g_{aw}^{(f)}(\theta) = g_w^{(h)}(\theta),$$

where h is as defined in (3) *and* (4).

Property 2.2. *Let f and h be the probability density functions of X and $-X$ respectively. Then*

$$g_{aw}^{(h)}(\theta) = g_{aw}^{(f)}(2\pi - \theta).$$

Property 2.3. *The alternate-wrapped density g_{aw} can be written as a mixture of the usual wrapped densities;*

$$g_{aw}(\theta) = p g_w^{(f_1)}(\theta) + (1-p) g_w^{(f_2)}(\theta), \quad 0 \le p \le 1,$$

where $g_w^{(f_i)}(\theta)$ is the usual wrapped density obtained by wrapping linear density $f_i(x)$, $i = 1,2$, respectively, as defined in (17) *in the Appendix.*

The proofs of the above properties are given in the Appendix.

Remark 2.1. *If X is symmetric about* 0*, then, g_{aw} is symmetric about* 0 *(or π.)*

Remark 2.2. *In alternate-wrapping, the density function $g(\theta)$ is not necessarily continuous at $\theta = 0$. However, the distribution function is continuous and satisfies the properties of a distribution function.*

Remark 2.3. *For an arbitrary density $f(x)$, if the density on the support $[0,\infty)$ is wrapped first in the clockwise direction and next in the anti-clockwise direction alternatively and the density on the support $(-\infty,0)$ in first anti-clockwise and next in clockwise direction alternatively, then a new wrapped circular density, say g_{aw}^- is obtained. This g_{aw}^- is the same as alternate-wrapped density g_{aw} obtained by wrapping $f(-x)$.*

Remark 2.4. *Characteristic Functions: The characteristic functions of the linear distribution and its usual wrapped distribution remain the same. But, in general, the characteristic function of the alternate-wrapped distribution is not equal to that of the linear distribution. However, this is true if the support of the density of X is a subset of $(-2\pi, 2\pi)$, as in this case the alternate-wrapped and the usual wrapped densities are the same.*

Distinction between usual and alternate-wrapping: Let X have the density function $f(x)$ and θ_w be a usual wrapped circular random variable with density function $g_w(\theta)$, obtained by wrapping the density f. Then, we have

$$g_w(\theta) = \sum_{k=-\infty}^{\infty} f(k2\pi + \theta),$$

and the distribution function of θ_w is given by

$$G_w(\alpha) = P(\theta_w \le \alpha) = \int_0^{\alpha} \left(\sum_{k=-\infty}^{\infty} f(k2\pi + \theta) \right) d\theta = \sum_{k=-\infty}^{\infty} \left(\int_0^{\alpha} f(k2\pi + \theta) d\theta \right),$$
$$0 \le \alpha \le 2\pi.$$

Consider the transformation

$$Y=\sum_{k=-\infty}^{\infty}(X-k2\pi)I_{k2\pi\leq X<(k+1)2\pi}. \qquad (5)$$

Then, the distribution function of Y is given by

$$H(\alpha)=P(Y\leq\alpha)=P\left(k(2\pi)\leq X<k(2\pi)+\alpha;\text{ for some } k=\cdots,-2,-1,0,1,2,\cdots\right).$$

This implies $H(\alpha)=\sum_{k=-\infty}^{\infty}\int_{k(2\pi)}^{k(2\pi)+\alpha}f(x)dx=\sum_{k=-\infty}^{\infty}\int_{0}^{\alpha}f(k2\pi+t)dt=P(\theta_w\leq\alpha)$. Hence, Y and θ_w are identically distributed. Hence, to generate an observation on θ_w, we generate X and obtain Y by using (5).

Now, on the other hand, let θ_{aw} be an alternate-wrapped circular random variable with density function $g_{aw}(\theta)$, obtained by alternate-wrapping of the density function $f(x)$. Then, we have $g_{aw}(\theta)=\sum_{k=-\infty}^{\infty}f(2k(2\pi)+\theta)+\sum_{k=-\infty}^{\infty}f((2k-1)(2\pi)+(2\pi-\theta))$, which implies

$$g_{aw}(\theta)=\sum_{k=-\infty}^{\infty}\left(f(2k(2\pi)+\theta)+f((2k)(2\pi)-\theta)\right).$$

The distribution function of θ_{aw} is given by

$$P(\theta_{aw}\leq\alpha)=\int_{0}^{\alpha}\left(\sum_{k=-\infty}^{\infty}\left(f(2k(2\pi)+\theta)+f((2k)(2\pi)-\theta)\right)\right)d\theta,$$

which gives

$$P(\theta_{aw}\leq\alpha)=\sum_{k=-\infty}^{\infty}\int_{4k\pi-\alpha}^{4k\pi+\alpha}f(x)dx.$$

Consider the transformation

$$Z=\sum_{k=-\infty}^{\infty}|X-4k\pi|I_{\{(2k-1)2\pi\leq X<(2k+1)2\pi\}}. \qquad (6)$$

The distribution function of Z is given by

$$M(\alpha)=P(Z\leq\alpha)=P\left(|X-4k\pi|\leq\alpha;\text{ for some } k=\cdots,-2,-1,0,1,2,\cdots\right),$$

which implies

$$M(\alpha)=\sum_{k=-\infty}^{\infty}\int_{4k\pi-\alpha}^{4k\pi+\alpha}f(x)dx=P(\theta_{aw}\leq\alpha).$$

Thus, θ_{aw} and Z are identically distributed. Thus, to generate observations on θ_{aw}, we generate X and use the transformation Z as given in (6). The distinctions between the usual wrapped variable Y and the alternate-wrapped variable Z can also be viewed in Figure 1.

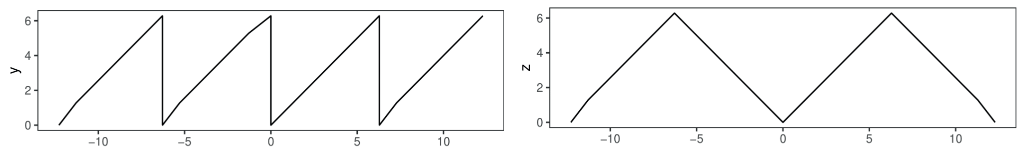


Figure 1. *Periodical behavior of usual and alternate-wrapped circular random variable Y and Z respectively.*

Remark 2.5. *Change of scale: Let X be a random variable with probability density function $f(x)$. Let $U = cX, c > 0$. For $0 < \alpha \leq 2\pi$, we have $G_{aw}^{U}(\alpha) = P(U \leq \alpha) = \sum_{k=-\infty}^{\infty} \int_{4k\pi-\alpha}^{4k\pi+\alpha} c^{-1} f(u/c)du$,*
which implies, $G_{aw}^{U}(\alpha) = \sum_{k=-\infty}^{\infty} \left(F\left(\frac{4k\pi+\alpha}{c}\right) - F\left(\frac{4k\pi-\alpha}{c}\right)\right)$, where, $F(.)$ is the distribution function of X.

3. Alternate-wrapped exponential (AWE) distribution

Let X follow the exponential distribution, then, we have

$$f(x) = \lambda e^{-\lambda x}, \quad \lambda > 0, x > 0.$$

A random variable θ is said to have an AWE distribution if, the circular density function of θ is given by (2)

$$g_{aw}(\theta) = \lambda e^{-\lambda\theta} + \sum_{m=1}^{\infty}\left(\lambda e^{-\lambda(4m\pi+\theta)} + \lambda e^{-\lambda(4m\pi-\theta)}\right), \ \theta \in [0, 2\pi).$$

That is

$$g_{aw}(\theta) = \frac{\lambda}{(1-e^{-4\pi\lambda})}\left(e^{-\lambda\theta} + e^{-\lambda(4\pi-\theta)}\right), \quad \lambda > 0, \theta \varepsilon [0, 2\pi). \tag{7}$$

For $\lambda = 1$ the usual, circular (Rao et al. (2013)), and 3D circular representations of (7) are given in Figure 2.

It is easy to verify (7) is a mixture of two WE circular densities, wherein the mixing proportion is a function of λ.

Remark 3.1. *$g_{aw}(\theta) = g_1(\theta)\rho(\lambda) + g_2(\theta)(1-\rho(\lambda))$ where $g_1(\theta) = \frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}$, $g_2(\theta) = \frac{\lambda e^{-\lambda(2\pi-\theta)}}{1-e^{-2\pi\lambda}}$ and $\rho(\lambda) = \frac{e^{2\pi\lambda}}{1+e^{2\pi\lambda}}$.*

Note that, $g_1(\theta)$ is the usual wrapped exponential density, see Jammalamadaka and Kozubowski (2004), and $g_2(\theta) = g_1(2\pi - \theta)$.

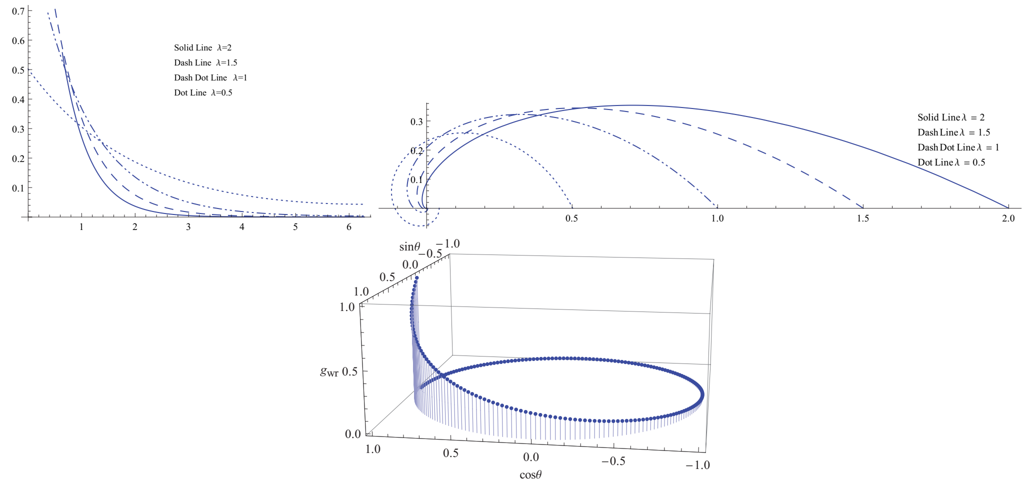


Figure 2. *Usual, circular, and 3D representations of the AWE density.*

3.1. Some properties of the AWE distribution

We obtain trigonometric moments, characteristic function, central trigonometric moments, see Mardia and Jupp (2000), and some other constants for the AWE distribution. All these expressions have been obtained by using Mathematica version 9. However, the same can also be obtained using Remark 3.1.

Trigonometric moments: To obtain the trigonometric moments, we first obtain the following.

$$\alpha_p = E(\cos p\theta) = \frac{\lambda(\lambda + p\operatorname{csch}(2\pi\lambda)\sin(2\pi p))}{\lambda^2 + p^2}, \tag{8}$$

and

$$\beta_p = E(\sin p\theta) = \frac{\lambda p\operatorname{csch}(2\pi\lambda)(\cosh(2\pi\lambda) - \cos(2\pi p))}{\lambda^2 + p^2}.$$

The p^{th} trigonometric moments of θ, denoted by ϕ_p, is the value of the characteristic function at an integer p and can be expressed in terms of α_p and β_p as follows

$$\phi_p = \alpha_p + i\beta_p = \rho_p e^{i\mu_p},$$

where $\rho_p = \sqrt{\alpha_p^2 + \beta_p^2}$ and $\mu_p = \arctan^*\left(\frac{\beta_p}{\alpha_p}\right)$, where the two-argument operator $\arctan^*$ is as defined in (1.3.5) (Jammalamadaka and SenGupta (2001)).

The mean resultant length ρ is given by $\rho = \rho_1 = \sqrt{\frac{\lambda^2\left(\lambda^2+\tanh^2(\pi\lambda)\right)}{(\lambda^2+1)^2}}$.

Characteristic function: Since, θ is a periodic circular random variable with period 2π, the characteristic function (corresponding to density (7)) is given by

$$\Phi_p = E(e^{ip\theta}) = E(e^{ip(2\pi+\theta)}) = E(\cos p\theta + i\sin p\theta),$$

which implies $\Phi_p = \frac{\lambda\left(\left(e^{4\pi\lambda}-1\right)\lambda + ip\left(e^{4\pi\lambda}-2e^{2\pi(\lambda+ip)}+1\right)\right)}{\left(e^{4\pi\lambda}-1\right)(\lambda^2+p^2)}, \quad p = 0, \pm1, \pm2, \cdots.$

Mean direction: The mean direction is $\mu = \mu_1 = \tan^{-1}\left(\frac{\tanh(\pi\lambda)}{\lambda}\right)$.

The circular variance: The circular variance is

$$V_0 = 1-\rho = 1-\sqrt{\frac{\lambda^2\left(\lambda^2+\tanh^2(\pi\lambda)\right)}{\left(\lambda^2+1\right)^2}},$$

where, $\rho = \rho_1$.

The circular standard deviation: is

$$\sigma_0 = \sqrt{-2\log(1-V_0)} = \sqrt{2}\sqrt{-\log\left(\sqrt{\frac{\lambda^2\left(\lambda^2+\tanh^2(\pi\lambda)\right)}{\left(\lambda^2+1\right)^2}}\right)}$$

Central trigonometric moments: The central trigonometric moments are given by

$$\bar{\phi}_p = E(e^{ip(\theta-\mu)}) = \bar{\alpha}_p + i\bar{\beta}_p,$$

where, μ is the mean direction. This implies

$$\bar{\phi}_p = \frac{\lambda\left(\left(e^{4\pi\lambda}-1\right)\lambda + ip\left(e^{4\pi\lambda}-2e^{2\pi(\lambda+ip)}+1\right)\right)e^{-i\mu p}}{\left(e^{4\pi\lambda}-1\right)(\lambda^2+p^2)}.$$

Since, $\bar{\phi}_p = E(e^{ip(\theta-\mu)}) = E(\cos p(\theta-\mu) + i\sin p(\theta-\mu))$, therefore,

$$E(\cos p(\theta-\mu)) = \bar{\alpha}_p$$
$$= \frac{\lambda\left(2e^{2\pi\lambda}p\sin((2\pi-\mu)p) + \left(e^{4\pi\lambda}+1\right)p\sin(\mu p) + \left(e^{4\pi\lambda}-1\right)\lambda\cos(\mu p)\right)}{\left(e^{4\pi\lambda}-1\right)(\lambda^2+p^2)},$$

and

$$E(\sin p(\theta-\mu)) = \bar{\beta}_p$$
$$= \frac{\lambda\left(\left(e^{4\pi\lambda}+1\right)p\cos(\mu p) - 2e^{2\pi\lambda}(\lambda\sinh(2\pi\lambda)\sin(\mu p) + p\cos((2\pi-\mu)p))\right)}{\left(e^{4\pi\lambda}-1\right)(\lambda^2+p^2)}.$$

The coefficient of skewness, $\zeta_1^0 = \frac{\bar{\beta}_2}{V_0^{3/2}}$, is given by

$$\zeta_1^0 = \frac{\lambda\left(2\left(e^{2\pi\lambda}-1\right)\cos(2\mu) - \left(e^{2\pi\lambda}+1\right)\lambda\sin(2\mu)\right)}{\left(e^{2\pi\lambda}+1\right)(\lambda^2+4)\left(1-\sqrt{\frac{\lambda^2\left(\lambda^2+\tanh^2(\pi\lambda)\right)}{\left(\lambda^2+1\right)^2}}\right)^{3/2}}.$$

The coefficient of kurtosis, $\zeta_2^0 = \frac{\bar{\alpha}_2-(1-V_0)^4}{V_0^2}$, is given by

$$\zeta_2^0 = \frac{\frac{\lambda(\lambda\cos(2\mu)+2\tanh(\pi\lambda)\sin(2\mu))}{\lambda^2+4} - \frac{\lambda^4\left(\lambda^2+\tanh^2(\pi\lambda)\right)^2}{(\lambda^2+1)^4}}{\left(\sqrt{\frac{\lambda^2\left(\lambda^2+\tanh^2(\pi\lambda)\right)}{(\lambda^2+1)^2}}-1\right)^2}.$$

3.2. Estimation of parameter

In this section, we obtain the MLE and the moment estimator of the parameter λ of AWE distribution.

MLE for AWE

Here we obtain the MLE of λ, the parameter of the AWE distribution. Let $\theta_1, \theta_2, \cdots, \theta_n$ be a random sample from the model (7). Then, the log-likelihood function is given by $\log L = n\log\lambda - n\log(1-e^{-4\pi\lambda}) + \sum_{i=1}^n \log\left(e^{-\lambda\theta_i} + e^{-\lambda(4\pi-\theta_i)}\right)$. The MLE of λ is obtained by solving the likelihood equation given by

$$\frac{\partial \log L}{\partial\lambda} = \frac{n}{\lambda} + \frac{4n\pi e^{-4\pi\lambda}}{1-e^{-4\pi\lambda}} - \sum_{i=1}^n \frac{4\pi e^{-\lambda(4\pi-\theta_i)} - \theta_i e^{-4\pi\lambda}}{\left(e^{-\lambda\theta_i} + e^{-\lambda(4\pi-\theta_i)}\right)} = 0. \tag{9}$$

Since (9) can not be solved analytically, we use a numerical method to obtain the MLE of λ. For this, we use the maxLik package in R which maximizes a function by using Newton-Raphson algorithm.

Moment estimator for AWE

Here we obtain the moment estimator for the parameter λ. Let $\theta_1, \cdots, \theta_n$ be a random sample from AWE distribution. To obtain the moment estimator, we equate $E(\cos\theta)$ to the mean of $\cos\theta_i's$. From (8) with $p=1$, we will have $\frac{\lambda^2}{\lambda^2+1} = \frac{1}{n}\sum_{i=1}^n \cos\theta_i = \bar{C}$ and hence the moment estimator is

$$\tilde{\lambda} = \sqrt{\frac{\bar{C}}{1-\bar{C}}}. \tag{10}$$

3.3. Simulation study

In this section, to evaluate the performance of the estimators, we carry out a simulation study. For large samples, consistency, asymptotic unbiasedness and asymptotic normality of MLE and moment estimator follow from the large sample theory. It is also well supported by the simulation study conducted. Here, we generate 10000 samples for finite sample sizes, 20 and 50 from the density (7) for different values of λ. Here instead of generating θ using (7), we generate Z as defined in (6). To obtain the MLE, we use the maxLik package in R since (9) can not be solved analytically. To obtain the moment estimators we use (10). Based on the simulation study, performances of estimators for small and moderate sample sizes are evaluated and results are reported in Tables 1 and 2.

Table 1. *Average values of bias, mean square error (MSE) and variance (var) for $\hat{\lambda}$ based on 10000 samples.*

	$\lambda = 0.5$			$\lambda = 1.0$		
n	bias($\hat{\lambda}$)	MSE($\hat{\lambda}$)	var($\hat{\lambda}$)	bias($\hat{\lambda}$)	MSE($\hat{\lambda}$)	var($\hat{\lambda}$)
20	0.017117	0.020537	0.019378	0.046972	0.060530	0.058539
50	0.010271	0.007256	0.007245	0.018571	0.024428	0.021548
	$\lambda = 2.0$			$\lambda = 4.0$		
20	0.118710	0.271947	0.237354	0.204775	1.042776	0.933834
50	0.040330	0.089551	0.085022	0.075890	0.351541	0.339197

Table 2. *Average values of bias, mean square error (MSE), and variance (var) for $\tilde{\lambda}$ based on 10000 samples.*

	$\lambda = 0.5$			$\lambda = 1.0$		
n	bias($\tilde{\lambda}$)	MSE($\tilde{\lambda}$)	var($\tilde{\lambda}$)	bias($\tilde{\lambda}$)	MSE($\tilde{\lambda}$)	var($\tilde{\lambda}$)
20	0.043820	0.051156	0.049241	0.038183	0.086479	0.085029
50	0.002439	0.023788	0.023785	0.015761	0.030834	0.030588
	$\lambda = 2.0$			$\lambda = 4.0$		
20	0.131001	0.309610	0.292478	0.347391	1.338005	1.217446
50	0.048435	0.097680	0.095343	0.127815	0.416024	0.399727

Based on the results obtained from Table 1 and 2 the following are some observations.
(i) The values of bias, MSE, and variance decrease as the sample size increases.
(ii) The bias, MSE, and variance values increase with the increase in the value of λ and decrease with the increase in sample size.
(iii) The MSE and variance of the MLE is lesser than that of the moment estimator for all values of λ.
(iv) The bias of MLE is higher than the moment estimator for $\lambda < 1$ and is lesser otherwise.
The box plots of the estimators and kernel density estimators of the densities have been plotted. Additionally, the histograms have been drawn for the estimated values and normal distribution has been fitted to those histograms which validate that the estimators are asymptotically normal. All these graphs and plots have been provided as supplementary material.

4. Alternate-wrapped normal (AWN) distribution

Let X have $N(\mu, \sigma^2)$, then the density function of X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}; -\infty < x < \infty, -\infty < \mu < \infty, \sigma^2 > 0. \tag{11}$$

From (1), the alternate-wrapped density function corresponding to (11) is given by

$$g_{aw}(\theta) = \frac{1}{\sigma\sqrt{2\pi}} \Bigg[e^{-\frac{1}{2\sigma^2}(\theta-\mu)^2} + e^{-\frac{1}{2\sigma^2}(-2\pi+\theta-\mu)^2} + e^{-\frac{1}{2\sigma^2}(-2\pi-\theta-\mu)^2} + \sum_{m=1}^{\infty} \left(e^{-\frac{1}{2\sigma^2}(4m\pi\pm\theta-\mu)^2} + e^{-\frac{1}{2\sigma^2}(-(4m+2)\pi\pm\theta-\mu)^2} \right) \Bigg]. \tag{12}$$

$$\begin{aligned} g_{aw}(\theta) = \frac{1}{\sigma\sqrt{2\pi}} \Bigg[& \exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right) + \exp\left(-\frac{(-2\pi-\theta-\mu)^2}{2\sigma^2}\right) + \exp\left(-\frac{(-2\pi+\theta-\mu)^2}{2\sigma^2}\right) \\ & + \exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right) \sum_{m=1}^{\infty} \exp\left(-\frac{16\pi^2 m^2}{2\sigma^2}\right) \sum_{m=1}^{\infty} \exp\left(-\frac{8\pi m(\theta-\mu)}{2\sigma^2}\right) \\ & + \exp\left(-\frac{(-\theta+\mu+2\pi)^2}{2\sigma^2}\right) \sum_{m=1}^{\infty} \exp\left(-\frac{16\pi^2 m^2}{2\sigma^2}\right) \sum_{m=1}^{\infty} \exp\left(-\frac{8\pi m(-\theta+\mu+2\pi)}{2\sigma^2}\right) \\ & + \exp\left(-\frac{(\theta+\mu)^2}{2\sigma^2}\right) \sum_{m=1}^{\infty} \exp\left(-\frac{16\pi^2 m^2}{2\sigma^2}\right) \sum_{m=1}^{\infty} \exp\left(-\frac{-8\pi m(\theta+\mu)}{2\sigma^2}\right) \\ & + \exp\left(-\frac{(\theta+\mu+2\pi)^2}{2\sigma^2}\right) \sum_{m=1}^{\infty} \exp\left(-\frac{16\pi^2 m^2}{2\sigma^2}\right) \sum_{m=1}^{\infty} \exp\left(-\frac{8\pi m(\theta+\mu+2\pi)}{2\sigma^2}\right) \Bigg]. \end{aligned}$$

$$\begin{aligned} g_{aw}(\theta) = \frac{1}{2\sigma\sqrt{2\pi}} \Bigg[& 2\exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right) + 2\exp\left(-\frac{(-2\pi-\theta-\mu)^2}{2\sigma^2}\right) \\ & + 2\exp\left(-\frac{(-2\pi+\theta-\mu)^2}{2\sigma^2}\right) + \sum_{m=1}^{\infty} \exp\left(-\frac{16\pi^2 m^2}{2\sigma^2}\right) \Bigg[\exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right) \\ & \left(\frac{1}{\exp\left(-\frac{2\pi(\theta-\mu)}{\sigma^2}\right) - 1}\right) + \exp\left(-\frac{(\theta+\mu)^2}{2\sigma^2}\right) \left(\frac{1}{\exp\left(-\frac{2\pi(\theta+\mu)}{\sigma^2}\right) - 1}\right) \\ & + \exp\left(-\frac{(2\pi+\theta+\mu)^2}{2\sigma^2}\right) \left(\frac{1}{\exp\left(\frac{2\pi(2\pi+\theta+\mu)}{\sigma^2}\right) - 1}\right) \\ & + \exp\left(-\frac{(2\pi-\theta+\mu)^2}{2\sigma^2}\right) \left(\frac{1}{\exp\left(\frac{2\pi(2\pi-\theta+\mu)}{\sigma^2}\right) - 1}\right) \Bigg] \Bigg]. \end{aligned} \tag{13}$$

Since, $\vartheta_3\left(0, e^{-\frac{8\pi^2}{\sigma^2}}\right) = 1 + 2\sum_{n=1}^{\infty} \exp\left(-8\pi^2/\sigma^2\right)^{n^2}$ is the elliptic theta function (please refer EllipticTheta 2022), (13) can also be written as

$$g_{aw}(\theta) = \frac{1}{2\sigma\sqrt{2\pi}} \Bigg[2\exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right) + 2\exp\left(-\frac{(-2\pi-\theta-\mu)^2}{2\sigma^2}\right)$$
$$+ 2\exp\left(-\frac{(-2\pi+\theta-\mu)^2}{2\sigma^2}\right) + \left(\vartheta_3\left(0, e^{-\frac{8\pi^2}{\sigma^2}}\right) - 1\right) \Bigg[\exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right)$$
$$\left(\frac{1}{\exp\left(-\frac{2\pi(\theta-\mu)}{\sigma^2}\right) - 1}\right) + \exp\left(-\frac{(\theta+\mu)^2}{2\sigma^2}\right) \left(\frac{1}{\exp\left(-\frac{2\pi(\theta+\mu)}{\sigma^2}\right) - 1}\right)$$
$$+ \exp\left(-\frac{(2\pi+\theta+\mu)^2}{2\sigma^2}\right) \left(\frac{1}{\exp\left(\frac{2\pi(2\pi+\theta+\mu)}{\sigma^2}\right) - 1}\right)$$
$$+ \exp\left(-\frac{(2\pi-\theta+\mu)^2}{2\sigma^2}\right) \left(\frac{1}{\exp\left(\frac{2\pi(2\pi-\theta+\mu)}{\sigma^2}\right) - 1}\right) \Bigg] \Bigg].$$

Remark 4.1. *If X follows $N(0,1)$, then $g_w(\theta) = g_{aw}(\theta)$.*

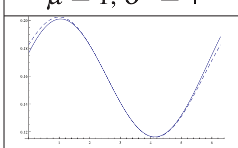
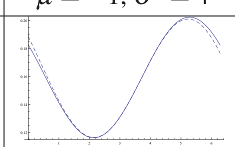
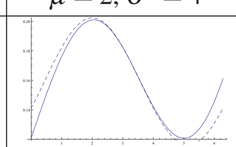
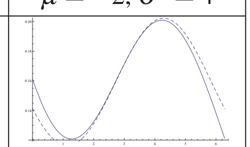
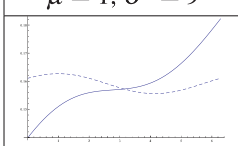
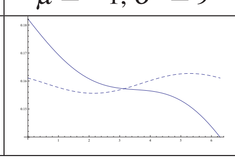
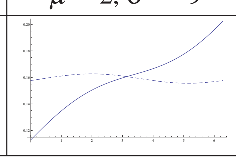
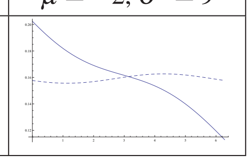
Property 4.1. *Let $g_{aw}(\theta, \mu)$ be the AWN density corresponding to $N(\mu, \sigma^2)$, then*

$$g_{aw}(\theta, \mu) = g_{aw}(2\pi - \theta, -\mu), \quad 0 < \theta < 2\pi.$$

The proof of this property is given in the Appendix.

The AWN and WN densities are plotted for different values of μ and σ^2 in Table 3.

Table 3. *AWN (solid line) and WN (dashed line) densities for different values of μ and σ^2.*

$\mu = 1$, $\sigma^2 = 4$	$\mu = -1$, $\sigma^2 = 4$	$\mu = 2$, $\sigma^2 = 4$	$\mu = -2$, $\sigma^2 = 4$
			
$\mu = 1$, $\sigma^2 = 9$	$\mu = -1$, $\sigma^2 = 9$	$\mu = 2$, $\sigma^2 = 9$	$\mu = -2$, $\sigma^2 = 9$
			

4.1. Estimation of parameters

In this section, we obtain the MLEs of μ and σ, the parameters of the AWN distribution. Let $\theta_1, \theta_2, \cdots, \theta_n$ be a random sample from the model (13). Then, the log-likelihood function is given by

$$\begin{aligned}\log L = & -\frac{n}{2}\log 2\pi - n\log\sigma - n\log 2 + \sum_{i=1}^{n}\log\Bigg[2\exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right) \\ & + 2\exp\left(-\frac{(-2\pi-\theta-\mu)^2}{2\sigma^2}\right) + 2\exp\left(-\frac{(-2\pi+\theta-\mu)^2}{2\sigma^2}\right) + \sum_{m=1}^{\infty}\exp\left(-\frac{16\pi^2m^2}{2\sigma^2}\right) \\ & \Bigg[\exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right)\left(\frac{1}{\exp\left(-\frac{2\pi(\theta-\mu)}{\sigma^2}\right)-1}\right) + \exp\left(-\frac{(\theta+\mu)^2}{2\sigma^2}\right) \\ & \left(\frac{1}{\exp\left(-\frac{2\pi(\theta+\mu)}{\sigma^2}\right)-1}\right) + \exp\left(-\frac{(2\pi+\theta+\mu)^2}{2\sigma^2}\right)\left(\frac{1}{\exp\left(\frac{2\pi(2\pi+\theta+\mu)}{\sigma^2}\right)-1}\right) \\ & + \exp\left(-\frac{(2\pi-\theta+\mu)^2}{2\sigma^2}\right)\left(\frac{1}{\exp\left(\frac{2\pi(2\pi-\theta+\mu)}{\sigma^2}\right)-1}\right)\Bigg]\Bigg]. \end{aligned} \tag{14}$$

To obtain the MLEs of μ and σ, we maximize the log-likelihood (14) numerically by using the maxLik package in *R*. For the computational purpose, we use the first three terms of the series $\sum_{m=1}^{\infty}\exp\left(-\frac{8\pi^2m^2}{\sigma^2}\right)$ yielding accuracy to at least five decimal places.

4.2. Simulation study

The simulation study is conducted by using different values of μ and σ for the sample sizes of 20 and 50 with 10000 replications. The AWN random variables are generated using (6). The MLEs of μ and σ are obtained by maximizing (14) using the maxLik package in R. The simulation results are reported as supplementary material. The following observations are made based on the simulation results.

1. For a given value of μ, the bias, MSE, and variance of $\hat{\mu}$ and $\hat{\sigma}$ increase as σ increases.

2. The bias, MSE, and variance of $\hat{\mu}$ and $\hat{\sigma}$ decrease with the increase in sample size.

For large samples, consistency, asymptotic unbiasedness, and asymptotic normality of MLEs follow from the large sample theory.

5. Data analysis

In this section, we fit the AWN and WN distributions to the data set of 506 cases of onset of lymphatic leukemia, reported in different months in the UK during 1946-1960 (Mardia and Jupp (2000)). For analysis purposes, the months are transformed into angles by assigning 30° sector to every month. The data are grouped, therefore all observations recorded every month are assigned to the midpoint of the interval, for example for the month of January, February, and March, the observations are assigned to the corresponding angles 15°, 45°, and 75°, respectively. By using the maxLik package in R we obtain, the MLEs (standard error) for both parameters of the AWN distribution as $\hat{\mu} = -2.3925$ (0.2010) and $\hat{\sigma} = 2.0465$ (0.1096). For the AWN model, μ and σ are the parameters and μ need not correspond to the mean direction of the corresponding AWN model. We also obtain the MLEs with standard errors for the parameters of WN distribution by using again the maxLik package in R: the values of the estimators are $\hat{\mu} = -2.7640$ (0.3181) and $\hat{\sigma} = 2.1440$ (0.1411). We apply the chi-square goodness of fit tests to both the AWN and WN models, by making six classes of the given data. We obtain the chi-square statistic values as 1.89 and 1.69 for the AWN and WN models respectively. The p-values of the statistics for both AWN and WN models are 0.8641 and 0.8901, respectively, which indicates that WN fits marginally better than AWN. Calculations of the chi-square statistics are given in supplementary material.

To evaluate the performance of the estimators under the AWN and WN models, we obtain AIC and BIC values. The AIC and BIC values for the AWN model are 1854.46 and 1862.91, which are very close to those of the WN model, 1853.31 and 1861.76, respectively. Based on the AIC and BIC values, we observe that the WN performs marginally better than the AWN, whereas based on the standard errors, the AWN performs marginally better than the WN.

Thus, one may conclude that for the considered data set, overall both the models perform almost the same.

6. Conclusion

In this paper, the concept of a novel wrapping technique called the alternate-wrapping technique has been introduced to generate circular models. Though the alternate-wrapped distributions are unable to retain some of the properties such as continuity at zero and the simplicity of obtaining characteristic functions, as are in the usual wrapping, they have some interesting properties, for example being expressible as a mixture of two usual wrapped distributions and others as indicated in the manuscript. The class of alternate-wrapped distributions widens the scope for research in circular models and data analysis. To enhance the class of circular distributions and for the circular data analysis, one can generate new alternate-wrapped versions of different distributions.

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Appendix

Property 2.2 Let f and h be the probability density functions of X and $-X$ respectively. Then

$$g_{aw}^{(h)}(\theta) = g_{aw}^{(f)}(2\pi - \theta).$$

Proof Let the density of X be $f(x)$. Now, if $Y = -X$ then, the density of Y can be given as $h(y) = f(-x)$. The total contribution at θ by the density h on the domain $\{y \geq 0\}$ is given by $h^+(\theta) = h(\theta) + h(4\pi - \theta) + h(4\pi + \theta) + h(8\pi - \theta) + h(8\pi + \theta) + \ldots$, which implies $h^+(\theta) = f(-\theta) + f(-4\pi + \theta) + f(-4\pi - \theta) + f(-8\pi + \theta) + f(-8\pi - \theta) + \ldots$. Hence,

$$\begin{aligned} h^+(2\pi - \theta) &= f(-2\pi + \theta) + f(-2\pi - \theta) + f(6\pi + \theta) + f(-6\pi - \theta) + f(-10\pi - \theta) + \ldots \\ &= f^-(\theta). \end{aligned} \tag{15}$$

The total contribution at θ by the density h on the domain $\{y < 0\}$ is $h^-(\theta) = h(-2\pi + \theta) + h(-2\pi - \theta) + h(-6\pi + \theta) + h(-6\pi - \theta) + h(-10\pi + \theta) + h(-10\pi - \theta) + \ldots$, which implies $h^-(\theta) = f(2\pi - \theta) + f(2\pi + \theta) + f(6\pi - \theta) + f(6\pi + \theta) + f(10\pi - \theta) + h(10\pi + \theta) + \ldots$. Hence,

$$\begin{aligned} h^-(2\pi - \theta) &= f(\theta) + f(4\pi - \theta) + f(4\pi + \theta) + f(8\pi - \theta) + f(8\pi + \theta) + f(12\pi - \theta) + \ldots \\ &= f^+(\theta). \end{aligned} \tag{16}$$

Thus, from (15) and (16), we have $h^{+}(2\pi-\theta)+h^{-}(2\pi-\theta)=f^{+}(\theta)+f^{-}(\theta)$. That is, $g_{aw}^{(h)}(\theta)=g_{aw}^{(f)}(2\pi-\theta)$ Hence the proof.

Property 2.3 The alternate-wrapped density g_{aw} can be written as a mixture of the usual wrapped densities;

$$g_{aw}(\theta)=pg_{w}^{(f_1)}(\theta)+(1-p)g_{w}^{(f_2)}(\theta), \quad 0\leq p\leq 1,$$

where $g_{w}^{(f_i)}(\theta)$ is the usual wrapped density obtained by wrapping linear density $f_i(x)$, $i=1,2$, respectively, as defined in (17).

Proof In the usual wrapping the entire density on positive support is wrapped in the anti-clockwise direction, starting from 0, and the entire density on negative support is wrapped in the clockwise direction, starting from 0. But in alternate-wrapping, the density on the positive support is alternatively wrapped, it is anti-clockwise wrapping for the intervals $(4\pi r, 4\pi r+2\pi)$ for $r=0,1,2,\dots$ and on the remaining intervals it is clockwise wrapping. Similarly, in alternate-wrapping, the density on the negative support is alternatively wrapped, it is clockwise wrapping on the intervals $(-4\pi r-2\pi, -4\pi r)$ for $r=0,1,2,\dots$, and on the remaining intervals it is anti-clockwise wrapping.
Let

$$A=\left\{\bigcup_{r=0}^{\infty}(4\pi r, 4\pi r+2\pi]\right\}\bigcup\left\{\bigcup_{r=0}^{\infty}(-4\pi r-2\pi, -4\pi r]\right\}.$$

That is, on the set A piece-wise wrapping directions for usual and alternate-wrapping are the same. On $R-A$, define a function f^*, by considering the reflection of the original function on each interval in $R-A$ of length 2π. On the interval $(k2\pi,\ (k+1)2\pi]$ for $k=1,3,5,\cdots$, we change the function by taking its reflection on $(2k+1)\pi$, the mid-point of the interval. Whereas, on the interval $(-(k+1)2\pi,\ -k2\pi]$ for $k=1,3,5,\cdots$, we change the function by taking its reflection on $-(2k+1)\pi$, the mid-point of the interval. The graphical representation of the set A together with the function f^* is given in Figure 4.

Let

$$f^*(t)=\begin{cases} f((2k+1)\pi-t) & \text{for}(k2\pi\leq t\leq (k+1)2\pi], k=1,3,5,\cdots \\ f(-(2k+1)\pi-t) & \text{for}(-(k+1)2\pi\leq t\leq -k2\pi], k=1,3,5,\cdots. \end{cases}$$

Let $p=\int_A f(x)dx$, if $p=1$ then the support for f will be a subset of A and in this case the alternate-wrapped and the usual wrapped densities will be the same. Let $0<p<1$, we note that

$$f_1(x)=\frac{f(x)I_{\{x\in A\}}}{p}, \quad f_2(x)=\frac{f^*(x)(1-I_{\{x\in A\}})}{1-p} \tag{17}$$

are density functions.

Let $g_{i,w}(\theta)$ be usual wrapped density corresponding to linear density $f_i(x)$, $i = 1,2$, respectively. Then, the alternate-wrapped density $g_{aw}(\theta)$ can be written as

$$g_{aw}(\theta) = p g_w^{(f_1)}(\theta) + (1-p) g_w^{(f_2)}(\theta), \quad 0 \leq p \leq 1,$$

We know that for an integrable function $h(x)$ over a finite interval (a,b), we have $\int_a^b h(x)dx = \int_a^b h^*(x)dx$, where $h^*(x) = h(a+b-x)$ is the reflection of the function $h(x)$ about $(a+b)/2$. Hence the result.

If $p = 0$, then we will have $g_{aw}(\theta) = g_w^{(f_2)}(\theta)$.

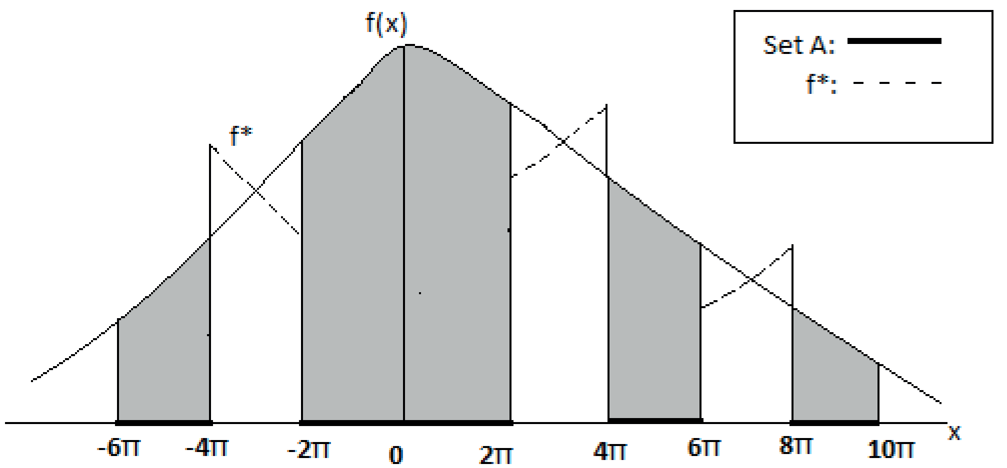

Figure 3. *Representation of the set A and the function f^*.*

Property 4.1 Let $g_{aw}(\theta,\mu)$ be the AWN density corresponding to $N(\mu,\sigma^2)$, then

$$g_{aw}(\theta,\mu) = g_{aw}(2\pi-\theta,-\mu), \quad 0 < \theta < 2\pi.$$

Proof Using (12), we can write

$$\begin{aligned} g_{aw}(2\pi-\theta,-\mu) = \frac{1}{\sigma\sqrt{2\pi}} \Bigg[& e^{-\frac{1}{2\sigma^2}(2\pi-\theta+\mu)^2} + e^{-\frac{1}{2\sigma^2}(-2\pi+2\pi-\theta+\mu)^2} + e^{-\frac{1}{2\sigma^2}(-2\pi-2\pi+\theta+\mu)^2} \\ & + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(4m\pi+2\pi-\theta+\mu)^2} + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(4m\pi-2\pi+\theta+\mu)^2} \\ & + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(-(4m+2)\pi+2\pi-\theta+\mu)^2} + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(-(4m+2)\pi-2\pi+\theta+\mu)^2} \Bigg], \end{aligned}$$

which implies

$$
\begin{aligned}
g_{aw}(2\pi-\theta,-\mu) = \frac{1}{\sigma\sqrt{2\pi}} \Bigg[& e^{-\frac{1}{2\sigma^2}(-2\pi+\theta-\mu)^2} + e^{-\frac{1}{2\sigma^2}(\theta-\mu)^2} + e^{-\frac{1}{2\sigma^2}(4\pi-\theta-\mu)^2} \\
& + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(-(4m+2)\pi+\theta-\mu)^2} + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}((4m-2)\pi+\theta+\mu)^2} \\
& + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(4m\pi+\theta-\mu)^2} + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}((4m+4)\pi-\theta-\mu)^2} \Bigg].
\end{aligned}
$$

This gives

$$
\begin{aligned}
g_{aw}(2\pi-\theta,-\mu) = \frac{1}{\sigma\sqrt{2\pi}} \Bigg[& e^{-\frac{1}{2\sigma^2}(-2\pi+\theta-\mu)^2} + e^{-\frac{1}{2\sigma^2}(\theta-\mu)^2} + e^{-\frac{1}{2\sigma^2}(-2\pi-\theta-\mu)^2} \\
& + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(-(4m+2)\pi+\theta-\mu)^2} + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(-(4m+2)\pi-\theta-\mu)^2} \\
& + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(4m\pi+\theta-\mu)^2} + \sum_{m=1}^{\infty} e^{-\frac{1}{2\sigma^2}(4m\pi-\theta-\mu)^2} \Bigg] = g_{aw}(\theta,\mu).
\end{aligned}
$$