SOLUCIÓ AL PROBLEMA PROPOSAT AL VOLUM 25 N. 2

PROBLEMA N. 89

It is well known that $(n-1)S \sim W_p(V, n-1)$. See, e.g., Anderson (1958, sections 3.3 and 7.2). This means that this seemingly non-central Wishart variate is, in fact, a central Wishart variate. A quick way to see this is the following. Write $(n-1)S = \sum_{i=1}^{n} (x_i - \overline{x})(x_i - \overline{x})' = X'MX$, with $X := (x_1, \ldots, x_n)$, $M := I_n - n^{-1} 1_n 1_n'$, 1_n being an $(n \times 1)$ vector with n unit elements.

As M is symmetric idempotent, its Schur decomposition is $M = TT'$, with $T'T = I_{n-1}$ and $T' 1_n = 0$. This yields then $(n-1)S = Y'Y$, with $Y' := X'T$. Write $Y' = (y_1, \ldots, y_{n-1})$. Clearly $\mathcal{D}(\mathrm{vec}\,Y') = \mathcal{D}(\mathrm{vec}\,X'T) = \mathcal{D}\left[(T' \otimes I_p(\mathrm{vec}\,X')\right] = (T' \otimes I_p)$ $\mathcal{D}(\mathrm{vec}\,X')(T \otimes I_p) = (T' \otimes I_p)(I_n \otimes V)(T \otimes I_p) = T'T \otimes V = I_{n-1} \otimes V$. The $n-1$ vectors $y_1, \ldots, y_{n-1}$ are seen to be uncorrelated. Because of normality they are independent. Further $EY' = (EX')T = \mu 1_n' T = 0$. Given the definition of the central Wishart we conclude that $(n-1)S \sim W_p(V, n-1)$.

It is also well known that $E\left[(n-1)S\right]^{-1} = (n-p-2)^{-1}V^{-1}$ so that $ES^{-1} = (n-1)(n-p-2)^{-1}V^{-1}$.

See, e.g. Legault-Giguère (1974, Lemma B6) or Neudecker (2001).

In a recent article Fang, Kollo & Parring (2000) give an approximation

$$E\,\mathrm{vec}\,S^{-1} = \mathrm{vec}\,V^{-1} + (2n)^{-1}\left(\mathrm{vec}\,\Pi \otimes I_{p^2}\right)' \mathrm{vec}\,B' + 0(n^{-1}),$$

with

$$\Pi := \left(I_{p^2} + K_{pp}\right)(V \otimes V)$$

and

$$B := (I_p \otimes K_{pp} \otimes I_p)\left[I_{p^2} \otimes \mathrm{vec}\,V^{-1} + \left(\mathrm{vec}\,V^{-1}\right) \otimes I_{p^2}\right]\left(V^{-1} \otimes V^{-1}\right)$$

(We added a tranposition sign to B in the result. It was apparently lost in the process.)

A little bit of straightforward algebra shows that

$$\left(\mathrm{vec}\,\Pi \otimes I_{p^2}\right)' \mathrm{vec}\,B' = 2\,(p\Pi)\,\mathrm{vec}\,V^{-1}.$$

Hence the approximation boils down to

$$ES^{-1} = n^{-1}(n+p+1)V^{-1} + 0(n-1).$$

References

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