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New L^2-type exponentiality tests

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Abstract

We introduce new consistent and scale-free goodness-of-fit tests for the exponential distribution based on the Puri-Rubin characterization. For the construction of test statistics we employ weighted L^2 distance between V-empirical Laplace transforms of random variables that appear in the characterization. We derive the asymptotic behaviour under the null hypothesis as well as under fixed alternatives. We compare our tests, in terms of the Bahadur efficiency, to the likelihood ratio test, as well as some recent characterization based goodness-of-fit tests for the exponential distribution. We also compare the power of our tests to the power of some recent and classical exponentiality tests. According to both criteria, our tests are shown to be strong and outperform most of their competitors.

MSC: 62G10, 62G20.

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1. Introduction

The exponential distribution is one of the most widely studied distributions in theoretical and applied statistics, and many models assume exponentiality of the data. For this reason, a great variety of goodness-of-fit tests, for the case of the exponential distribution, have been proposed in the literature.

The classical approach is to use the time-honoured goodness-of-fit tests based on an empirical distribution function, such as Kolmogorov-Smirnov, Cramer-von Mises, Anderson-Darling, applied to the case of the exponential distributions. The alternative approach is to use tests specifically designed for testing exponentiality. These test statistics are mainly based on empirical counterparts of certain special properties of the exponential distribution. Some of the tests employ properties related to different integral transforms such as: characteristic functions (see e.g. Henze, 1992, Henze and Meintanis, 2002b, Henze and Meintanis, 2005); Laplace transforms (see e.g. Henze

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and Meintanis, 2002a, Klar, 2003, Meintanis, Nikitin and Tchirina, 2007); and other integral transforms (see e.g. Klar, 2005, Meintanis, 2008). Other tests exploit properties such as maximal correlations (see Grané and Fortiana, 2009, Grané and Fortiana, 2011, Strzalkowska-Kominiak and Grané, 2017), entropy (see Alizadeh Noughabi and Arghami, 2011), etc.

Among the various properties, those that characterize the distribution stand out. The simple form of the exponential distribution give rise to many equidistribution type characterizations. The equality in distribution can be expressed in many ways (equality of distribution functions, densities, integral transforms, etc.), and hence is suitable for building different types of test statistics. Such tests have become very popular in recent times, as they are proven to be rather efficient. Tests that use U-empirical and V-empirical distribution functions, of integral-type (integrated difference) and supremum-type, can be found in Nikitin and Volkova (2010), Volkova (2015), Jovanović et al.(2015), Milošević and Obradović (2016b), Milošević (2016), Nikitin and Volkova (2016). A class of weighted integral-type tests that uses U-empirical Laplace transforms is presented in Milošević and Obradović (2016a).

Motivated by the power and efficiency of those tests, we create a similar test based on an equidistribution characterization. The test statistics, measuring the distance between two V-empirical Laplace transforms of the random variables that appear in the characterization, are, for the first time, of weighted L^2-type. This guarantees the consistency of the test against all alternatives.

The paper is organized as follows. In Section 2 we introduce the test statistics and derive their asymptotic properties, both under the null and the alternative hypotheses. In Section 3 we calculate the approximate Bahadur slope of our tests, for different close alternatives, and inspect the impact of the tuning parameter to the efficiencies of the test. We also compare the proposed tests to their recent competitors via approximate local relative Bahadur efficiency. In Section 4 we conduct a power study. We obtain empirical powers of the tests, against different common alternatives, and compare them to some recent and classical exponentiality tests. We also apply an algorithm for data driven selection of the tuning parameter and obtain the corresponding powers in the small sample case. Real data applications are presented in Section 5, while the proofs, the datasets, and the code can be found in the appendices.

2. Test statistic

Consider the following characterization by Puri and Rubin (1970).

Characterization 2.1. *Let X_1 and X_2 be two independent copies of a random variable X with pdf $f(x)$. Then X and $|X_1 - X_2|$ have the same distribution, if and only if for some $\lambda > 0$, $f(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.*

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (i.i.d.) non-negative random variables with an unknown absolutely continuous distribution function F. We consider the transformed sample $Y_i = \widehat{\lambda}_n X_i,\ i = 1,2..,n.$, where $\widehat{\lambda}_n$ is the reciprocal sample mean. For testing the null hypothesis $H_0:\ F(x) = 1 - e^{-\lambda x},\ \lambda > 0$, in view of the characterization 2.1, we propose the following family of test statistics, depending on the tuning parameter $a > 0$:

$$M_{n,a}(\widehat{\lambda}_n) = \int_0^\infty \Big(L_n^{(1)}(t) - L_n^{(2)}(t)\Big)^2 e^{-at}dt, \tag{1}$$

where

$$L_n^{(1)}(t) = \frac{1}{n}\sum_{i_1=1}^{n} e^{-tY_{i_1}}$$

$$L_n^{(2)}(t) = \frac{1}{n^2}\sum_{i_1,i_2=1}^{n} e^{-t|Y_{i_1}-Y_{i_2}|}$$

are V-empirical Laplace transforms of Y_1 and $|Y_1 - Y_2|$ respectively.

In order to explore the asymptotic properties we rewrite (1) as

$$\begin{aligned}
M_{n,a}(\widehat{\lambda}_n) &= \int_0^\infty \left(\frac{1}{n}\sum_{i_1=1}^{n} e^{-tX_{i_1}\widehat{\lambda}_n} - \frac{1}{n^2}\sum_{i_1,i_2=1}^{n} e^{-t|X_{i_1}-X_{i_2}|\widehat{\lambda}_n}\right)^2 e^{-at}dt \\
&= \frac{1}{n^4}\int_0^\infty \sum_{i_1,i_2,i_3,i_4} \Big(e^{-tX_{i_1}\widehat{\lambda}_n} - e^{-t|X_{i_1}-X_{i_2}|\widehat{\lambda}_n}\Big)\Big(e^{-tX_{i_3}\widehat{\lambda}_n} - e^{-t|X_{i_3}-X_{i_4}|\widehat{\lambda}_n}\Big) e^{-at}dt \\
&= \frac{1}{n^4}\sum_{i_1,i_2,i_3,i_4}\int_0^\infty g(X_{i_1},X_{i_2},t;\widehat{\lambda}_n)g(X_{i_3},X_{i_4},t;\widehat{\lambda}_n)e^{-at}dt \\
&= \frac{1}{n^4}\sum_{i_1,i_2,i_3,i_4} h(X_{i_1},X_{i_2},X_{i_3},X_{i_4},a;\widehat{\lambda}_n),
\end{aligned}$$

where $\widehat{\lambda}_n = \overline{X}_n^{-1}$ is a consistent estimator of λ and

$$h(X_1,X_2,X_3,X_4,a;\widehat{\lambda}_n) = \frac{1}{4!}\sum_{\pi(4)}\int_0^\infty g(X_{i_1},X_{i_2},t;\widehat{\lambda}_n)g(X_{i_3},X_{i_4},t;\widehat{\lambda}_n)e^{-at}dt,$$

with $\pi(4)$ being the set of all 4! permutations of the numbers $1,2,3,4$.

Let us first focus on $M_{n,a}(\lambda)$, for a fixed $\lambda > 0$. Notice that $M_{n,a}(\lambda)$ is a V-statistic with kernel h. Moreover, under the null hypothesis, its distribution does not depend on λ, so we may assume $\lambda = 1$. It is easy to show that its first projection on a basic observation is equal to zero. After some calculations, one can obtain its second projection given by

$$\begin{aligned}\widetilde{h}_2(x,y,a) &= E(h(X_1,X_2,X_3,X_4,a)|X_1 = x, X_2 = y)\\ &= -\frac{1}{2}+\frac{1}{3}(e^{-x}+e^{-y})+\frac{1}{6}e^{a-x-y}\mathrm{Ei}(-a)\Big(a(e^x-2)(e^y-2)-e^x-e^y+4\Big)\\ &+\frac{1}{6}e^{-a-x-y}\Big(\mathrm{Ei}(a)(4a+e^x+e^y-4)-(\mathrm{Ei}(a+x)(4(a+x-1)+e^y)\\ &+\mathrm{Ei}(a+y)(4(a+y-1)+e^x)-4(a+x+y-1)\mathrm{Ei}(a+x+y))\Big)+\frac{1}{6(a+x+y)},\end{aligned}$$

where $\mathrm{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}dt$ is the exponential integral. The function $\widetilde{h}_2$ is non-constant for any $a > 0$. Hence, kernel h is degenerate with degree 2.

Since kernel h is bounded and degenerate, from the theorem for the asymptotic distribution of U-statistics with degenerate kernels (Korolyuk and Borovskikh, 1994, Corollary 4.4.2), and the Hoeffding representation of V-statistics, we get that, $M_{n,a}(1)$, being a V-statistic of degree 2, has the following asymptotic distribution

$$nM_{n,a}(1) \xrightarrow{d} 6\sum_{k=1}^{\infty}\delta_k W_k^2, \tag{2}$$

where $\{\delta_k\}$ are the eigenvalues of the integral operator $\mathcal{M}_a$ defined by

$$\mathcal{M}_a q(x) = \int_0^{+\infty}\widetilde{h}_2(x,y,a)q(y)dF(y), \tag{3}$$

and $\{W_k\}$ is the sequence of i.i.d. standard Gaussian random variables.

Our statistic $M_{n,a}(\widehat{\lambda}_n)$ can be rewritten as

$$\begin{aligned}M_{n,a}(\widehat{\lambda}_n) &= \int_0^{\infty}\left(\frac{1}{n^2}\sum_{i_1,i_2=1}^{n}g(X_{i_1},X_{i_2},t,a;\widehat{\lambda}_n)\right)^2 e^{-at}dt\\ &= \int_0^{\infty}V_n(\widehat{\lambda}_n)^2e^{-at}dt.\end{aligned}$$

Here $V_n(\widehat{\lambda}_n)$ is a V-statistic of order 2 with an estimated parameter, and kernel $g(X_{i_1},X_{i_2},t,a;\widehat{\lambda}_n)$.

Since the function $g(x_1,x_2,t,a;\gamma)$ is continuously differentiable with respect to γ at the point $\gamma=\lambda$, the mean-value theorem gives

$$V_n(\widehat{\lambda}_n)=V_n(\lambda)+(\widehat{\lambda}_n-\lambda)\frac{\partial V_n(\gamma)}{\partial\gamma}|_{\gamma=\lambda^*},$$

for some λ^* between λ and $\widehat{\lambda}_n$.

From the Law of large numbers for V-statistics (Serfling, 2009, 6.4.2.), the partial derivative $\frac{\partial V_n(\gamma)}{\partial\gamma}$ converges to

$$E\left(t|X_1-X_2|e^{-t|X_1-X_2|\gamma}-tX_1e^{-tX_1\gamma}\right)=0.$$

Since $\sqrt{n}(\widehat{\lambda}_n-\lambda)$ is stochastically bounded, it follows that statistics $\sqrt{n}V_n(\widehat{\lambda}_n)$ and $\sqrt{n}V_n(1)$ are asymptotically equally distributed. Therefore, $nM_{n,a}(\widehat{\lambda}_n)$ and $nM_{n,a}(1)$ will have the same limiting distribution. We summarize this in the following theorem.

Theorem 2.2. *Let $X_1,\dots,X_n$ be an i.i.d. sample with distribution function $F(x)=1-e^{-\lambda x}$ for some $\lambda>0$. Then*

$$nM_{n,a}(\widehat{\lambda}_n)\overset{d}{\to}6\sum_{k=1}^{\infty}\delta_kW_k^2, \tag{4}$$

where $\{\delta_k\}$ are the eigenvalues of the integral operator $\mathcal{M}_a$ defined in (3)*, and $\{W_k\}$ is the sequence of i.i.d standard Gaussian random variables.*

2.1. Limiting distribution under fixed alternative

Now we consider the asymptotic behaviour of our statistics $M_{n,a}$ under a fixed alternative with finite expectation μ. Here, it is also easy to show that the first projection of kernel h,

$$h_1(s,a)=E(h(X_1,X_2,X_3,X_4,a;\mu)|X_1=s)$$

is a non-constant function, hence the kernel is non-degenerate. Therefore, the limiting distribution will differ from the null case. We present this in Theorem 2.3, where, for brevity, we introduce the following notation: $\boldsymbol{x}=(x_1,x_2,x_3,x_4)$; $\boldsymbol{G}(\boldsymbol{x})=\prod_{i=1}^4G(x_i)$; $h'(\boldsymbol{x},a;\mu)=\frac{\partial h(\boldsymbol{x},a;\gamma)}{\partial\gamma}\Big|_{\gamma=\mu}$.

Theorem 2.3. *Let $X_1,\dots,X_n$ be an i.i.d. sample from an alternative distribution with distribution function G. Then*

$$\sqrt{n}(M_{n,a}(\widehat{\mu})-\Delta)\overset{d}{\to}\mathcal{N}(0,\Sigma),$$

where $\widehat{\mu} = \overline{X}_n$, $\Delta = E(M_{n,a}(\mu))$, *and*

$$\Sigma = 16Var(h_1(X_1,a)) + \Big(\int\limits_{(R^+)^4} h'(\boldsymbol{x},a;\mu)d\boldsymbol{G}(\boldsymbol{x})\Big)^2 Var(X_1) + 8\Big(\int\limits_{(R^+)^4} x_1 h(\boldsymbol{x},a;\mu)d\boldsymbol{G}(\boldsymbol{x}) - \int\limits_{R^+} x_1 dG(x_1) \int\limits_{(R^+)^4} h(\boldsymbol{x},a;\mu)d\boldsymbol{G}(\boldsymbol{x})\Big). \quad (5)$$

Proof. See Appendix A. ∎

3. Local approximate Bahadur efficiency

One way to compare tests is to calculate their relative Bahadur efficiency. We briefly present it here. For more details we refer to Bahadur (1971) and Nikitin (1995).

For two tests with the same null and alternative hypotheses, $H_0 : \theta \in \Theta_0$ and $H_1 : \theta \in \Theta_1$, the asymptotic relative Bahadur efficiency is defined as the ratio of sample sizes needed to reach the same test power, when the level of significance approaches zero. For two sequences of test statistics, it can be expressed as the ratio of Bahadur exact slopes, functions proportional to the exponential rates of the decrease of their sizes, for the increasing number of observations and a fixed alternative. The calculation of these slopes depends on large deviation functions which are often hard to obtain. For this reason, in many situations, the tests are compared using the approximate Bahadur efficiency, which is shown to be a good approximation in the local case (when $\theta \to \partial\Theta_0$).

Suppose that $T_n = T_n(X_1,\dots,X_n)$ is a test statistic with its large values being significant. Let the limiting distribution function of T_n, under H_0, be F_T, whose tail behaviour is given by $\log(1-F_T(t)) = -\frac{a_T t^2}{2}(1+o(1))$, where a_T is a positive real number, and $o(1) \to 0$ as $t \to \infty$. Suppose also that the limit in probability $\lim_{n\to\infty} T_n/\sqrt{n} = b_T(\theta) > 0$ exists for $\theta \in \Theta_1$. Then the relative approximate Bahadur efficiency of T_n, with respect to another test statistic V_n (whose large values are significant), is

$$e^*_{T,V} = \frac{c^*_T(\theta)}{c^*_V(\theta)},$$

where $c^*_T(\theta) = a_T b^2_T(\theta)$ i $c^*_V(\theta) = a_V b^2_V(\theta)$ are approximate Bahadur slopes of T_n and V_n, respectively.

We may suppose, without loss of generality, that $\Theta_0 = \{0\}$. Consequently, the approximate local relative Bahadur efficiency is given by

$$e^*_{T,V} = \lim_{\theta\to 0} e^*_{T,V}(\theta).$$

Let $\mathcal{G} = \{G(x,\theta),\ \theta > 0\}$ be a family of alternative distribution functions with finite expectations, such that $G(x,\theta) = 1 - e^{-\lambda x}$, for some $\lambda > 0$, if and only if $\theta = 0$, and the regularity conditions for V-statistics with weakly degenerate kernels from (Nikitin and Peaucelle, 2004, Assumptions WD) are satisfied.

The logarithmic tail behaviour of the limiting distribution of $M_{n,a}(\widehat{\lambda}_n)$, under the null hypothesis, is derived in the following lemma.

Lemma 3.1. *For the statistic $M_{n,a}(\widehat{\lambda}_n)$ and a given alternative density $g(x,\theta)$ from $\mathcal{G}$, the Bahadur approximate slope satisfies the relation $c_M(\theta) \sim \frac{b_M(\theta)}{6\delta_1}$, where $b_M(\theta)$ is the limit in P_θ probability of $M_{n,a}(\widehat{\lambda}_n)$, and δ_1 is the largest eigenvalue of the sequence $\{\delta_k\}$ from* (2)*.*

Proof. See Appendix A.

The limit in probability of our test statistic, under a close alternative, can be derived using the following lemma.

Lemma 3.2. *For a given alternative density $g(x;\theta)$ whose distribution belongs to $\mathcal{G}$, we have that the limit in probability of the statistic $M_{n,a}(\widehat{\lambda}_n)$ is*

$$b_M(\theta) = 6\int_0^\infty \int_0^\infty \widetilde{h}_2(x,y)f(x)f(y)dxdy \cdot \theta^2 + o(\theta^2), \theta \to 0,$$

where $f(x) = \frac{\partial}{\partial\theta} g(x;\theta)|_{\theta=0}$.

Proof. See Appendix A.

To calculate the efficiency one needs to find δ_1, the largest eigenvalue. Since we can not obtain it analytically, we use the following approximation, introduced in Božin et al. (2018).

It can be shown that δ_1 is the limit of the sequence of the largest eigenvalues of linear operators defined by $(m+1)\times(m+1)$ matrices $M^{(m)} = ||m_{i,j}^{(m)}||$, $0 \le i \le m, 0 \le j \le m$, where

$$m_{i,j}^{(m)} = \widetilde{h}_2\left(\frac{Bi}{m}, \frac{Bj}{m}\right)\sqrt{e^{\frac{B(i)}{m}} - e^{\frac{B(i+1)}{m}}} \cdot \sqrt{e^{\frac{B(j)}{m}} - e^{\frac{B(j+1)}{m}}} \cdot \frac{1}{1-e^{-B}}, \tag{6}$$

when m tends to infinity and $F(B)$ approaches 1.

In Table 1, we present the largest eigenvalues for a =0.5, 1, 2 and 5, obtained using (6) with $m = 4500$ and $B = 10$.

Table 1: *Approximate eigenvalues of $\mathcal{M}_a$.*

a	0.5	1	2	5
δ_1	$1.32\cdot10^{-2}$	$5.32\cdot10^{-3}$	$1.73\cdot10^{-3}$	$2.80\cdot10^{-4}$

3.1. Efficiencies with respect to likelihood ratio tests

Lacking a theoretical upper bound, the approximate Bahadur slopes are often compared (see e.g. Meintanis et al., 2007) to the approximate Bahadur slopes of the likelihood ratio tests (LRT), which are known to be optimal parametric tests in terms of Bahadur efficiency. Hence, we may consider the approximate relative Bahadur efficiencies against the LRT as a sort of "absolute" local approximate Bahadur efficiencies. We calculate it for the following alternatives:

- a Weibull distribution with density

$$g(x,\theta)=e^{-x^{1+\theta}}(1+\theta)x^{\theta},\theta>0,x\geq 0; \quad (7)$$

- a Gamma distribution with density

$$g(x,\theta)=\frac{x^{\theta}e^{-x}}{\Gamma(\theta+1)},\theta>0,x\geq 0; \quad (8)$$

- a Linear failure rate (LFR) distribution with density

$$g(x,\theta)=e^{-x-\theta\frac{x^2}{2}}(1+\theta x),\theta>0,x\geq 0; \quad (9)$$

- a mixture of exponential distributions with negative weights (EMNW(β)) with density (see Jevremovic (1991))

$$g(x,\theta)=(1+\theta)e^{-x}-\theta\beta e^{-\beta x},\theta\in\left(0,\frac{1}{\beta-1}\right],x\geq 0; \quad (10)$$

It is easy to show that all densities given above belong to the family $\mathcal{G}$.

The efficiencies, as functions of the tuning parameter a, are shown on Figure 1.

We can notice that the local efficiencies range from reasonable to high, and for some values of a they are very high. Also, their behaviour with respect to the tuning parameter a is very different. In the cases of Weibull and Linear failure rate alternatives, they are increasing functions of a, while in the Gamma case, the function is decreasing. In the case of EMNW(3), the efficiencies increase up to a certain point and then decrease.

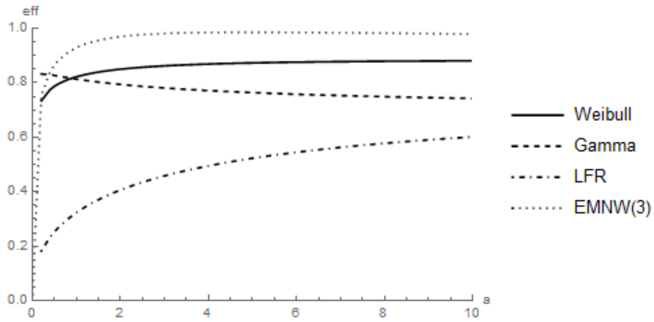

Figure 1: *Local approximate Bahadur efficiencies w.r.t. LRT.*

3.2. Comparison of efficiencies

In this section, we calculate the local approximate Bahadur relative efficiency of our tests against some recent, characterization based integral-type tests, for the previously mentioned alternatives.

The characterizations are of the equidistribution type and take the following form.

Let $X_1,\dots,X_{\max(m,p)}$ be i.i.d with d.f. F, $\omega_1 : R^m \mapsto R^1$ and $\omega_2 : R^p \mapsto R^1$ two sample functions. Then the following relation holds

$$\omega_1(X_1,\dots,X_m) \stackrel{d}{=} \omega_2(X_1,\dots,X_p)$$

if and only if $F(x) = 1 - e^{-\lambda x}$, for some $\lambda > 0$.

Notice that the Puri-Rubin characterization 2.1 is an example of such characterizations.

The first class of competitor tests consists of the integral-type tests with test statistic

$$I_n = \int_0^\infty \Big(G_n^{(1)}(t) - G_n^{(2)}(t)\Big) dF_n(t),$$

where $G_n^{(1)}(t)$ and $G_n^{(2)}(t)$ are V-empirical distribution functions of ω_1 and ω_2, respectively and F_n is the empirical distribution function.

In particular, we consider the following integral-type test statistics:

- $I_{n,k}^{(1)}$, proposed in Jovanović et al. (2015), based on the Arnold and Villasenor characterization, where $\omega_1(X_1,\dots,X_k) = \max(X_1,\dots,X_k)$ and $\omega_2(X_1,\dots,X_k) = X_1 + \frac{X_2}{2} + \cdots \frac{X_k}{k}$ (see Arnold and Villasenor, 2013, Milošević and Obradović, 2016c);

- $I_n^{(2)}$, proposed in Milošević and Obradović (2016b), based on the Milošević-Obradović characterization, where $\omega_1(X_1,X_2) = \max(X_1.X_2)$ and $\omega_2(X_1,X_2,X_3) = \min(X_1,X_2)+X_3$ (see Milošević and Obradović, 2016c);
- $I_n^{(3)}$, proposed in Milošević (2016), based on the Obradović characterization, where $\omega_1(X_1,X_2,X_3) = \max(X_1,X_2,X_3)$ and $\omega_2(X_1,X_2,X_3,X_4) = X_1+\mathrm{med}(X_2,X_3,X_4)$ (see Obradović, 2015);
- $I_n^{(4)}$, proposed in Volkova (2015), based on the Yanev-Chakraborty characterization, where $\omega_1(X_1,X_2,X_3) = \max(X_1,X_2,X_3)$ and $\omega_2(X_1,X_2,X_3) = \frac{X_1}{3}+\max(X_2,X_3)$ (see Yanev and Chakraborty, 2013).

We also consider integral-type tests of the form

$$J_{n,a} = \int_0^\infty \Big(L_n^{(1)}(t) - L_n^{(2)}(t)\Big)\overline{X}_n e^{-at}\,dt, \tag{11}$$

where $L_n^{(1)}(t)$ and $L_n^{(2)}(t)$ are V-empirical Laplace transforms of ω_1 and ω_2, respectively. This approach has been originally proposed in Milošević and Obradović (2016a). There, particular cases of Desu characterization, with $\omega_1(X_1) = X_1$ and $\omega_2(X_1,X_2) = 2\min(X_1,X_2)$, and Puri-Rubin characterization were examined. We denote the corresponding test statistics with $J_{n,a}^{\mathcal{D}}$ and $J_{n,a}^{\mathcal{P}}$, respectively. The results are presented in Table 2. We can notice that in most cases tests that employ V-empirical Laplace transforms are more efficient than those based on V-empirical distribution functions. On the other hand, new tests are comparable with $J_{n,a}^{\mathcal{P}}$ and more efficient than $J_{n,a}^{\mathcal{D}}$.

4. Power study

In this section we compare the empirical powers of our tests with those of some common competitors. We choose the values of the tuning parameter to be 0.5, 1, 2 and 5. We also consider the limiting case when a tends to infinity. The expression for this limiting statistic is given in the following theorem.

Theorem 4.1. *For fixed n, we have*

$$\lim_{a\to\infty} a^3 M_{n,a}(\hat{\lambda}_n) = 2\left(\frac{1}{n^2}\sum_{i,j=1}^{n}|Y_i - Y_j| - \overline{Y}_n\right)^2,$$

where $Y_i = \hat{\lambda}_n X_i,\ i = 1,2,\ldots,n.$

Table 2: *Relative Bahadur efficiency of $M_{n,a}$ with respect to its competitors.*

$I_{n,2}^{(1)}$	*Weibull*	1.27	1.33	1.37	1.42
	Gamma	1.14	1.13	1.10	1.06
	LFR	2.44	3.13	3.93	5.08
	EMNW(3)	1.25	1.34	1.40	1.42
$I_{n,3}^{(1)}$	*Weibull*	1.19	1.24	1.28	1.32
	Gamma	1.17	1.15	1.12	1.09
	LFR	1.59	2.04	2.56	3.31
	EMNW(3)	1.08	1.17	1.22	1.23
$I_n^{(2)}$	*Weibull*	1.05	1.10	1.14	1.17
	Gamma	1.04	1.02	1.00	0.97
	LFR	1.22	1.56	1.96	2.53
	EMNW(3)	1.02	1.10	1.15	1.17
$I_n^{(3)}$	*Weibull*	1.06	1.10	1.14	1.18
	Gamma	1.18	1.16	1.14	1.10
	LFR	0.82	1.05	1.32	1.71
	EMNW(3)	0.94	1.02	1.06	1.08
$I_n^{(4)}$	*Weibull*	1.21	1.27	1.31	1.35
	Gamma	1.30	1.28	1.25	1.21
	LFR	1.23	1.57	1.98	2.56
	EMNW(3)	1.04	1.12	1.16	1.18
$J_{n,a}^{\mathfrak{P}}$	*Weibull*	0.97	0.97	1.01	1.00
	Gamma	0.98	0.99	1.00	1.02
	LFR	0.97	0.93	0.91	0.93
	EMNW(3)	0.97	0.98	0.99	1.00
$J_{n,a}^{\mathfrak{D}}$	*Weibull*	1.00	0.95	0.93	0.95
	Gamma	2.16	1.64	1.33	1.13
	LFR	1.17	1.07	1.01	0.99
	EMNW(3)	1.42	1.18	1.06	0.99

Proof. See Appendix A.

As competitor tests we use the following tests, listed in Henze and Meintanis (2005), Milošević and Obradović (2016a) and Torabi, Montazeri and Grané (2018):

- the test based on mean density (see Epps and Pulley, 1986):
  $EP_n = \sqrt{48n}\left(\frac{1}{n}\sum_{j=1}^{n} e^{-Y_j} - \frac{1}{2}\right)$, where $Y_j = \frac{X_j}{\overline{X}_n}$;
- the tests based on the mean residual life function (see Baringhaus and Henze, 2000a):

$$\overline{KS}_n = \sqrt{n}\sup_{t\geq 0}\left|\frac{1}{n}\sum_{j=1}^{n}\min(Y_j,t) - \frac{1}{n}\sum_{j=1}^{n} I\{Y_j \leq t\}\right|;$$

$$\overline{CM}_n = n\int_0^\infty \left(\frac{1}{n}\sum_{j=1}^{n}\min(Y_j,t) - \frac{1}{n}\sum_{j=1}^{n} I\{Y_j \leq t\}\right)^2 e^{-t}dt;$$

- the Cramer-von Mises test: $\omega_n^2 = \int_0^\infty (F_n(x) - (1-e^{-x}))^2 e^{-x}dx$;
- the Kolmogorov-Smirnov test: $KS_n = \sup_{x\geq 0}|F_n(x) - (1-e^{-x})|$;
- the test based on the integrated distribution function (see Klar, 2001): $KL_{n,a} = na^3\int_0^\infty (\psi_n(t) - \psi(t))^2 e^{-at}dt$, where $\psi(t) = \int_t^\infty (1-F(x))dx = e^{-t}$ and $\psi_n(t) = \int_t^\infty (1-F_n(x))dx$;
- the test based on spacings and Gini index (see D'Agostino and Stephens, 1986): $S_n = \sum_{j=1}^{n-1} U_j$, where $U_j = \frac{\sum_{i=1}^{j} D_i}{\sum_{i=1}^{n} X_i}$ and $D_j = (n+1-j)(X_{(j)} - X_{(j-1)})$;
- the score test of Cox and Oakes (1984): $CO_n = n + \sum_{j=1}^{n}(1-Y_j)\log Y_j$;
- the test of Milošević and Obradović: $J_{n,a}^{\mathcal{D}}$ and $J_{n,a}^{\mathcal{P}}$ from (11);
- the tests based on discrepancy measure (see Torabi et al., 2018): $H_n^{(k)} = \frac{1}{n}\sum_{j=1}^{n} h_k\left(\frac{1+F_0(\frac{X_j}{\overline{X}_n})}{1+F_n(X_j)}\right)$, where $h_1(x) = (e^{x-1}-x)I_{[0,1]}(x) + \sqrt[3]{|x^3-1|}I_{[1,\infty)}(x)$ and $h_2(x) = (e^{x-1}-x)I_{[0,1]}(x) + \frac{(x-1)^2}{(x+1)^2}I_{[1,\infty)}(x)$;
- the test based on maximal correlations (see Fortiana and Grané, 2003): $Q_n = \frac{s_n}{\overline{X}_n}\rho^+(F_n, F_0)$, where s_n^2 is sample variance and $\rho^+(F_1,F_2)$ is Hoeffding maximum correlation.

The Monte Carlo study is done for the small sample size $n = 20$, and a moderate sample size $n = 50$, with $N = 10000$ replicates, for the level of significance $\alpha = 0.05$ and the following alternative distributions:

- a Weibull $W(\theta)$ distribution with density (7);
- a Gamma $\Gamma(\theta)$ distribution with density (8);

- a half-normal HN distribution with density

$$g(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}, x \geq 0;$$

- a uniform U distribution with density

$$g(x) = 1, 0 \leq x \leq 1;$$

- a Chen's $CH(\theta)$ distribution with density

$$g(x,\theta) = 2\theta x^{\theta-1} e^{x^{\theta}-2(1-e^{x^{\theta}})}, x \geq 0;$$

- a linear failure rate $LF(\theta)$ distribution with density (9);
- a modified extreme value $EV(\theta)$ distributions with density

$$g(x,\theta) = \frac{1}{\theta} e^{\frac{1-e^x}{\theta}+x}, x \geq 0.$$

The powers are presented in Tables 3 and 4.

Table 3: *Percentage of rejected hypotheses for n = 20.*

Alt.	$Exp(1)$	$W(1.4)$	$\Gamma(2)$	HN	U	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EV(1.5)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$
EP	5	36	48	21	66	63	15	84	28	42	45	25	67	20	64
$\overline{KS}$	5	35	46	24	72	47	18	79	32	44	48	28	55	22	6
$\overline{CM}$	5	35	47	22	70	61	16	83	30	43	47	27	66	21	63
ω^2	5	34	47	21	66	61	14	79	28	41	43	33	62	23	65
KS	5	28	40	18	52	56	13	67	24	34	35	30	58	20	56
KL	5	29	44	16	61	77	11	76	23	34	37	35	66	21	63
S	5	35	46	21	70	63	15	84	29	42	46	24	67	19	62
CO	5	37	54	19	50	**80**	13	81	25	37	37	33	60	25	72
$J_{n,1}^{\mathcal{D}}$	5	42	64	20	45	15	15	15	29	40	36	47	32	28	72
$J_{n,5}^{\mathcal{D}}$	5	48	64	28	70	20	21	21	36	52	53	33	57	24	70
$J_{n,1}^{\mathcal{P}}$	5	49	65	29	73	21	22	21	38	51	54	34	41	24	68
$J_{n,5}^{\mathcal{P}}$	5	48	62	**32**	79	23	23	23	**41**	**56**	58	27	59	21	65
$H_n^{(1)}$	5	49	60	31	78	0	**24**	**91**	40	55	23	33	0	30	74
$H_n^{(2)}$	5	6	10	2	18	79	2	29	4	7	8	8	**71**	4	20
Q_n	5	32	38	23	**86**	43	17	85	30	42	54	18	61	15	50
$M_{n,0.5}$	5	46	66	25	64	19	18	19	35	49	46	**57**	1	**39**	**81**
$M_{n,1}$	5	49	66	28	72	21	21	21	38	52	53	51	2	37	**81**
$M_{n,2}$	5	**50**	**67**	31	75	22	23	23	40	55	56	45	6	37	**81**
$M_{n,5}$	5	48	62	**32**	80	22	23	24	40	**56**	58	42	21	33	80
$M_{n,\infty}$	5	47	59	31	81	23	23	23	40	56	**59**	33	51	26	73

Table 4: Percentage of rejected hypotheses for $n = 50$.

Alt.	$Exp(1)$	$W(1.4)$	$\Gamma(2)$	HN	U	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EV(1.5)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$
EP	5	80	91	54	98	94	38	**100**	69	87	90	45	**95**	39	97
$\overline{KS}$	5	71	86	50	99	90	36	**100**	65	82	88	62	92	43	96
$\overline{CM}$	5	77	90	53	99	94	37	**100**	69	87	90	65	**95**	44	97
ω^2	5	75	90	48	98	95	32	**100**	64	83	86	76	94	52	98
KS	5	64	83	39	93	92	26	98	53	72	75	71	91	46	95
KL	5	72	93	37	97	**99**	23	**100**	54	75	79	92	94	66	99
S	5	79	90	54	99	94	38	**100**	69	87	90	47	**95**	39	97
CO	5	82	96	45	91	**99**	30	**100**	60	80	78	66	92	55	99
$J_{n,1}^{\mathcal{D}}$	5	78	96	36	76	23	24	23	51	71	64	93	64	72	**100**
$J_{n,5}^{\mathcal{D}}$	5	86	**97**	55	97	41	40	40	72	89	89	70	90	55	**100**
$J_{n,1}^{\mathcal{P}}$	5	85	96	54	97	38	38	38	70	87	87	77	78	58	99
$J_{n,5}^{\mathcal{P}}$	5	86	96	63	99	46	46	45	77	91	93	58	92	47	98
$H_n^{(1)}$	5	**88**	94	**65**	99	0	**50**	**100**	**79**	**92**	94	51	0	50	98
$H_n^{(2)}$	5	37	62	13	78	98	7	94	24	44	46	47	**95**	23	87
Q_n	5	73	79	59	**100**	77	47	**100**	74	89	**96**	26	93	25	86
$M_{n,0.5}$	5	84	**97**	48	95	34	33	33	65	83	81	**94**	36	**77**	**100**
$M_{n,1}$	5	85	**97**	54	97	38	38	38	69	87	86	89	50	72	**100**
$M_{n,2}$	5	86	96	57	98	41	41	41	73	89	90	83	65	67	**100**
$M_{n,5}$	5	87	96	63	99	45	45	45	76	91	93	71	80	59	99
$M_{n,\infty}$	5	84	94	63	99	47	46	46	78	**92**	94	53	92	48	98

It can be noticed that our tests have good empirical sizes and their power ranges from reasonable to high. In the majority of cases, our tests are either the most powerful or their power is very close to the one of the most powerful competitor.

4.1. On a data-dependent choice of the tuning parameter

The powers of the proposed tests depend on the values of the tuning parameter a. Therefore, a well-chosen value of a would help underpin making the right decision. However, since the "right" value of a is rather different for various alternatives, a general conclusion on which a is most suitable in practice, can not be made. Hence, in what follows, we present an algorithm for a data driven selection of the tuning parameter, proposed initially by Allison and Santana (2015):

1. fix a grid of positive values of $a, (a_1, \dots, a_k)$;
2. obtain a bootstrap sample $\boldsymbol{X}_n^*$ from the empirical distribution function of $\boldsymbol{X}_n$;
3. determine the value of the test statistic $M_{n,a_i}, i = 1, \dots, k$, for the obtained sample;
4. repeat steps 2 and 3 B times and obtain series of values of test statistics for every a, $M^*_{j,a_i}, i = 1, \dots, k, j = 1, \dots, B$;

5. determine the empirical power of the test for every a, i.e.

$$\hat{P}_{a_i} = \frac{1}{B}\sum_{j=1}^{B} \mathrm{I}\{M_{j,a_i} \geq \check{C}_{n,a_i}(\alpha)\}, i = 1, \ldots, k,$$

where $\mathrm{I}\{\cdot\}$ is the indicator function;
6. for the next calculation $\hat{a} = \underset{a\in\{a_1,\ldots,a_k\}}{argmax}\ \hat{P}_a$ will be used.

The critical value $\check{C}_{n,\hat{a}}$ is determined using the Monte Carlo procedure with N_1 replicates. Then, the empirical power of the test is determined based on the new sample from the alternative distribution

$$p = \frac{1}{N_1}\sum_{i=1}^{N_1} \mathrm{I}\{M_{n,\hat{a}} \geq \check{C}_{n,\hat{a}}(\alpha)\}.$$

The previously described procedure is repeated N times and the average value is taken as the estimated power:

$$\widetilde{P} = \frac{1}{N}\sum_{i=1}^{N} p_i.$$

The code of this algorithm is provided in Appendix C.

The results are presented in Tables 5 and 6. The numbers in the parentheses represent the percentage of times that each value of a equals the estimated optimal one. It is important to note that these bootstrap powers are comparable to the maximum achievable power for the tests calculated over a grid of values of the tuning parameter.

Table 5: *Percentage of rejected samples for different value of a, $n = 20$, $\alpha = 0.05$.*

	0.5	1	2	5	$\hat{a}$
$W(1.4)$	46(**50**)	49(12)	50(15)	48(23)	48
$\Gamma(2)$	66 (**63**)	65(12)	65(10)	63(15)	65
HN	25(**35**)	28(14)	30(17)	32(34)	29
U	64(20)	72(9)	75(21)	80 (**50**)	75
$CH(0.5)$	19(**37**)	21(15)	22(17)	22(31)	21
$CH(1)$	18(**35**)	21(15)	23(16)	23(34)	21
$CH(1.5)$	19(**35**)	20(11)	20(20)	24(34)	21
$LF(2)$	35(33)	37(12)	38(20)	41 (**35**)	38
$LF(4)$	49(**35**)	53(14)	54(16)	54(**35**)	52
$EW(1.5)$	46(24)	53(12)	56(20)	58(**44**)	54
$LN(0.8)$	57(**92**)	51(3)	45(4)	42(1)	56
$LN(1.5)$	2(13)	3(2)	6(2)	20(**83**)	17
$DL(1)$	39(**73**)	37(8)	37(10)	33(9)	38
$DL(1.5)$	82(**71**)	81(6)	82(12)	79(11)	82

Table 6: *Percentage of rejected samples for different value of a, $n = 50$, $\alpha = 0.05$.*

	0.5	1	2	5	$\hat{a}$
$W(1.4)$	84(**43**)	86(19)	86(16)	87(22)	85
$\Gamma(2)$	97(**68**)	97(15)	96(11)	95(6)	97
HN	48(21)	53(13)	57(23)	62(**43**)	57
U	95(31)	97(12)	98(20)	99(**37**)	98
$CH(0.5)$	34(19)	37(11)	41(20)	44(**50**)	41
$CH(1)$	33(18)	37(13)	41(18)	46(**51**)	41
$CH(1.5)$	33(18)	37(13)	42(19)	44(**50**)	41
$LF(2)$	65(20)	69(12)	74(24)	76(**44**)	72
$LF(4)$	83(25)	86(16)	89(20)	91(**39**)	88
$EW(1.5)$	81(17)	87(13)	89(22)	93(**48**)	89
$LN(0.8)$	95(**92**)	89(6)	82(2)	70 (0)	94
$LN(1.5)$	38(2)	50(1)	64(4)	79(**93**)	78
$DL(1)$	77(**81**)	72(11)	68(4)	60(4)	67
$DL(1.5)$	100(**84**)	100(11)	100(4)	99(1)	100

5. Real data examples

In this section we apply our tests to three real datasets. The first dataset represents inter-occurrence times of fatal accidents to British-registered passenger aircraft, 1946-1963, measured in number of days and listed in the order of their occurrence in time (see Pyke, 1965).

The second dataset represents failure times for right rear breaks on D9G-66A Caterpillar tractors (see Barlow and Campo (1975)). The third dataset represents failure and running times (1000 cycles) of a sample of 30 units of a larger electrical system (see Meeker and Escobar (2014)). The third set was also analysed in Shakeel et al. (2016). The datasets are given in Tables 8-10 of Appendix B, while their empirical and theoretical density, cumulative distribution function, Q-Q and P-P plots, are shown in Figures 2-4. The figures suggest that the exponential distribution provides a good fit for the first dataset, unlike for the remaining two.

In Table 7 we present, for all three datasets, the p-values of our test with data driven selection of the tuning parameter, as well as for $M_{n,\infty}$. For comparison purposes, we also include some exponentiality tests that were shown to have good power performance in Tables 3 and 4.

We can see that our tests confirm the conclusions suggested by the plots 2-4. While the competitor tests mostly point to the same decisions, it is worth noting that, at the 5% level of significance, few of them fail to reject the null hypothesis for the third dataset.

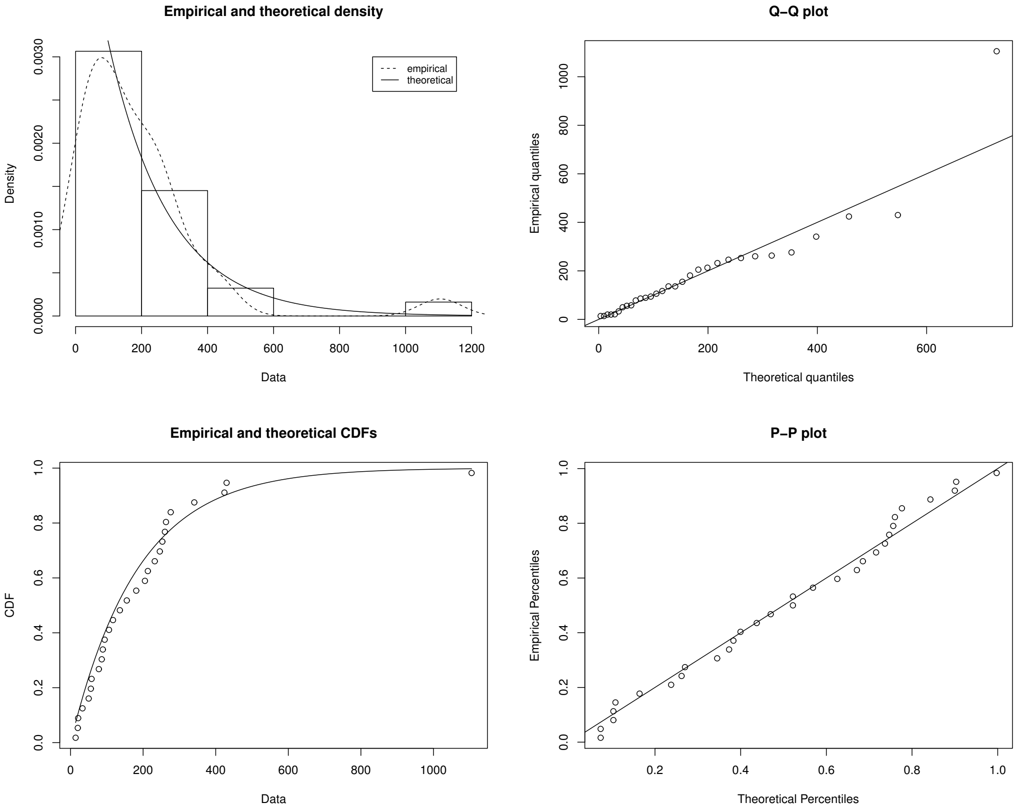

Figure 2: *Plots for dataset 1.*

Table 7: *p-values for three datasets.*

Stat.	EP	ω^2	CO	$H_n^{(1)}$	$H_n^{(2)}$	$J_{n,1}^{\mathcal{P}}$	$M_{n,\hat{a}}$	$M_{n,\infty}$
Dataset 1	0.4037	0.9103	0.4907	0.6062	0.8737	0.3708	0.4902	0.8917
Dataset 2	0	0	0	0	0.0005	0	0	0
Dataset 3	0.0279	0.0059	0.1536	0.0420	0.0940	0.0163	0.0092	0.0074

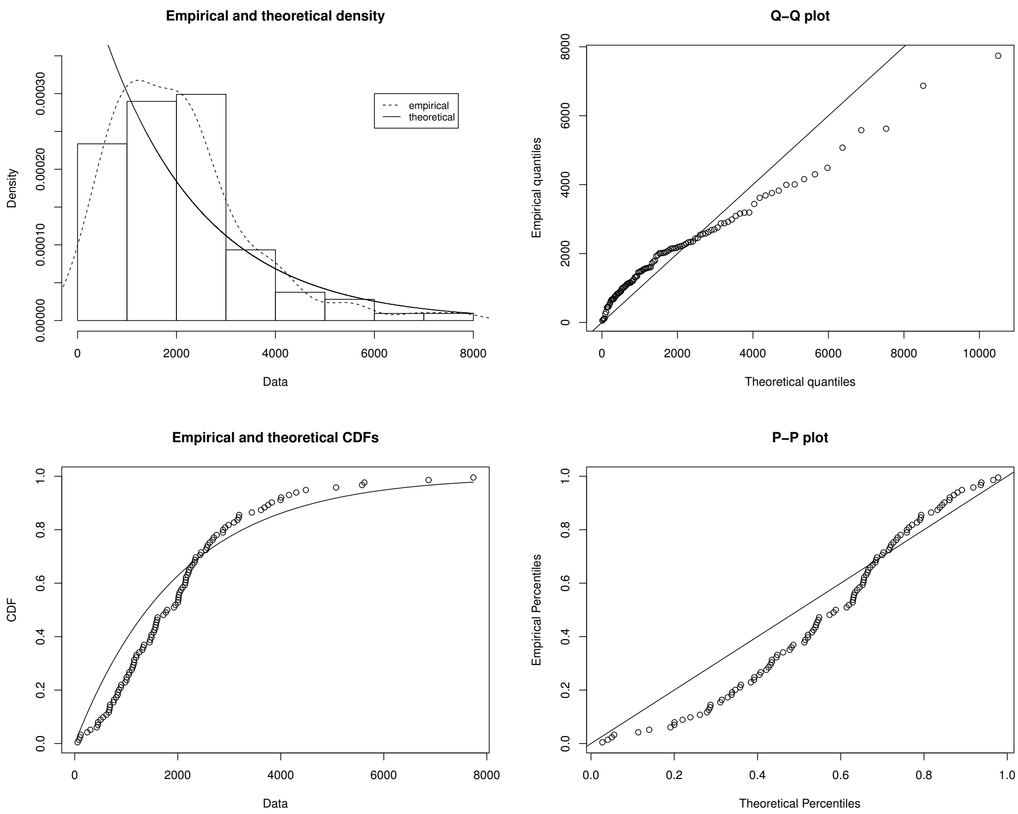

Figure 3: *Plots for dataset 2.*

6. Conclusion

In this paper we propose new consistent scale-free exponentiality tests based on the Puri-Rubin characterization. The proposed tests are shown to be very efficient in the Bahadur sense. Moreover, in the small sample case, the tests have reasonable to high empirical powers. They also outperform many recent competitor tests in terms of both efficiency and power. The quality of their performance is confirmed on two real data examples. This makes them attractive for use in practice.

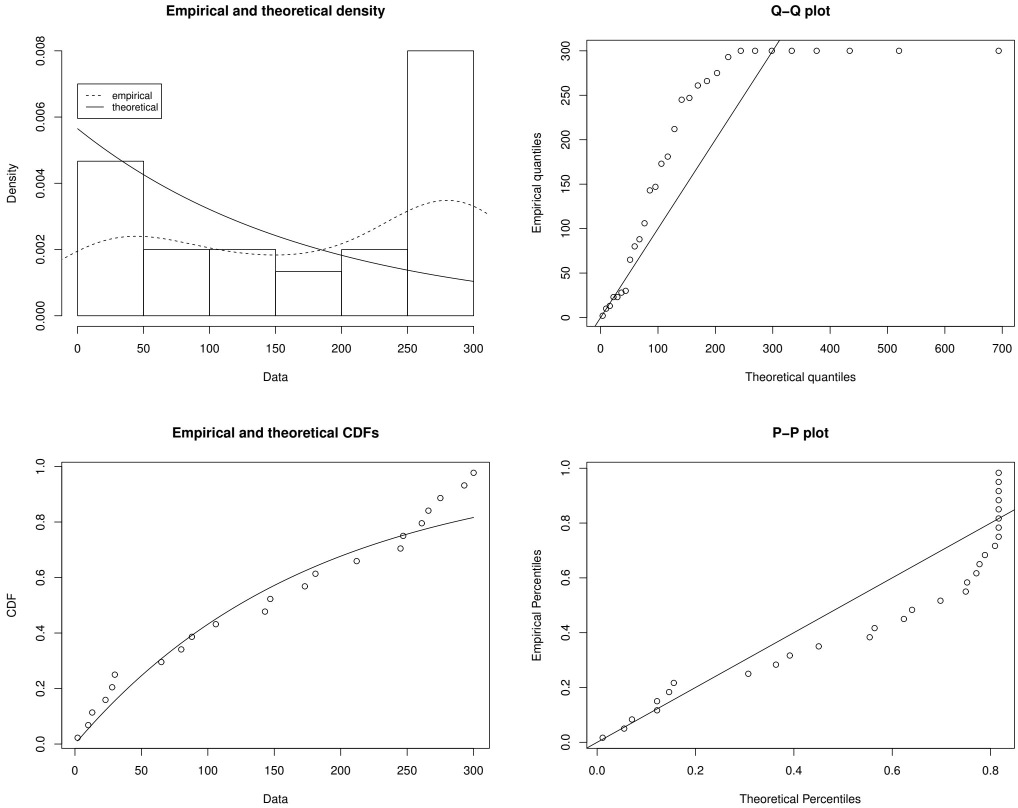

Figure 4: *Plots for dataset 3.*

Appendix A - Proofs

Proof of Theorem 2.3. Since the kernel h is non-degenerate, from the theorem for V-statistics with non-degenerate kernels (Korolyuk and Borovskikh, 1994, Theorem 4.2.5), it follows

$$\sqrt{n}(M_{n,a}(\mu)-\Delta)\stackrel{d}{\to}\mathcal{N}(0,16Var(h_1(X_1,a))).$$

As the function $h(x_1,x_2,x_3,x_4,a;\gamma)$ is continuously differentiable with respect to γ at the point $\gamma=\mu$, the mean-value theorem gives

$$\sqrt{n}(M_{n,a}(\widehat{\mu})-\Delta(\mu))=\sqrt{n}(M_{n,a}(\mu)-\Delta(\mu))+\sqrt{n}(\widehat{\mu}-\mu)\frac{\partial M_{n,a}(\gamma)}{\partial\gamma}|_{\gamma=\mu^*},$$

for some μ^* between μ and $\widehat{\mu}$.

Using the Law of large numbers for V-statistics, the Slutsky theorem, and the fact that the limit distribution of $\sqrt{n}(M_{n,a}(\mu)-\Delta,\widehat{\mu}-\mu)$ is two dimensional normal, it follows that $\sqrt{n}(M_{n,a}(\widehat{\mu})-\Delta(\mu))$ will converge in distribution to zero mean normal random variable, with the variance equal to

$$16Var(h_1(X_1,a))+\lim_{n\to\infty}E\left(\frac{\partial M_{n,a}(\gamma)}{\partial\gamma}\right)^2 Var(\sqrt{n}\widehat{\mu})+2\lim_{n\to\infty}Cov(\sqrt{n}M_{n,a}(\mu),\sqrt{n}\widehat{\mu}).$$

Calculating the limits, we obtain (5). ■

Proof of Lemma 3.1. Using the result of Zolotarev (1961), the logarithmic tail behaviour of limiting distribution function of $\widetilde{M}_{n,a}(\widehat{\lambda}_n)=\sqrt{nM_{n,a}(\widehat{\lambda}_n)}$ is

$$\log(1-F_{\widetilde{M}_a}(t))=-\frac{t^2}{12\delta_1}+o(t^2),\ \ t\to\infty.$$

Therefore, $a_{\widetilde{M}_a}=\frac{1}{6\delta_1}$. The limit in probability P_θ of $\widetilde{M}_{n,a}(\widehat{\lambda}_n)/\sqrt{n}$ is

$$b_{\widetilde{M}_a}=\sqrt{b_M(\theta)}.$$

Inserting this into the expression for Bahadur slope completes the proof. ■

Proof of Lemma 3.2. For brevity, denote $\boldsymbol{x}=(x_1,x_2,x_3,x_4)$ and $\boldsymbol{G}(\boldsymbol{x};\theta)=\prod_{i=1}^4 G(x_i;\theta)$. Since $\overline{X}_n$ converges almost surely to its expected value $\mu(\theta)$, using the Law of large numbers for V-statistics with estimated parameters (see Iverson and Randles, 1989), $M_{n,a}(\widehat{\lambda}_n)$ converges to

$$\begin{aligned}b_M(\theta)&=E_\theta(h(\boldsymbol{X},a;\mu(\theta)))\\&=\int\limits_{(R^+)^4}\Big(\frac{\mu(\theta)}{x_1+x_3+a\mu(\theta)}-\frac{\mu(\theta)}{x_3+|x_1-x_2|+a\mu(\theta)}\\&\quad-\frac{\mu(\theta)}{x_1+|x_3-x_4|+a\mu(\theta)}+\frac{\mu(\theta)}{|x_1-x_2|+|x_3-x_4|+a\mu(\theta)}\Big)d\boldsymbol{G}(\boldsymbol{x};\theta).\end{aligned}$$

We may assume that $\mu(0)=1$ due to the scale freeness of the test statistic under the null hypothesis. After some calculations we get that $b_M'(0)=0$. Further,

$$b''(0)=\int\limits_{(R^+)^4}h(\boldsymbol{x},a;1)\frac{\partial^2}{\partial\theta^2}d\boldsymbol{G}(\boldsymbol{x},0)=6\int\limits_{(R^+)^2}\widetilde{h}_2(x,y)f(x)f(y)dxdy.$$

Expanding $b_M(\theta)$ into the Maclaurin series we complete the proof. ■

Proof of Theorem 4.1. Denote $g(t) = \left(L_n^{(1)}(t) - L_n^{(2)}(t)\right)^2$. Then the test statistic can be expressed as $M_{n,a}(\hat{\lambda}_n) = \int_0^\infty g(t)e^{-at}dt$. The Maclaurin expansion of $g(t)$ is

$$g(t) = t^2\left(\frac{1}{n^2}\sum_{i,j=1}^{n}|Y_i - Y_j| - \overline{Y}_n\right)^2 + t^3\left(\frac{1}{n^2}\sum_{i,j=1}^{n}|Y_i - Y_j| - \overline{Y}_n\right)\left(\overline{Y}_n - \frac{1}{n^2}\sum_{i,j=1}^{n}(Y_i - Y_j)^2\right)$$
$$+ \frac{t^4}{4}\left(\overline{Y}_n - \frac{1}{n^2}\sum_{i,j=1}^{n}(Y_i - Y_j)^2\right) + o(t^4).$$

Using an Abelian theorem for the Laplace transform from (Widder, 1946, Chapter 5.2.) (see also from Baringhaus, Gürtler and Henze, 2000b, Proposition 1.1), and

$$\lim_{s\to\infty}\Gamma(4)s^3\int_0^s g(t)dt = 2\left(\frac{1}{n^2}\sum_{i,j=1}^{n}|Y_i - Y_j| - \overline{Y}_n\right)^2,$$

follows the statement of the theorem. ∎

Appendix B - Datasets

Table 8: *Dataset 1: inter-occurrence times of fatal accidents.*

20	106	14	78	94	20	21	136	56	232	89
33	181	424	14	430	155	205	117	253	86	260
213	58	276	263	246	341	1105	50	136		

Table 9: *Dataset 2: failure times for right rear breaks.*

56	83	104	116	244	305	429	452	453	503	552
614	661	673	683	685	753	763	806	834	838	862
897	904	981	1007	1008	1049	1060	1107	1125	1141	1153
1154	1193	1201	1253	1313	1329	1347	1454	1464	1490	1491
1532	1549	1568	1574	1586	1599	1608	1723	1769	1795	1927
1957	2005	2010	2016	2022	2037	2065	2096	2139	2150	2156
2160	2190	2210	2220	2248	2285	2325	2337	2351	2437	2454
2546	2565	2584	2624	2675	2701	2755	2877	2879	2922	2986
3092	3160	3185	3191	3439	3617	3685	3756	3826	3995	4007
4159	4300	4487	5074	5579	5623	6869	7739			

Table 10: *Dataset 3: failure and running times of units of an electrical system .*

275	13	147	23	181	30	65	10	300	173
106	300	300	212	300	300	300	2	261	293
88	247	28	143	300	23	300	80	245	266

Appendix C - Code

- `expTestL2puri` is a C-function which calculates the value of statistic $M_{n,a}$;
- `bootstrapStat` is an R-function which calculates statistics $M_{n,a}$ based on bootstrapped resamples from the initial sample;
- `optimal_a` is an R-function which implements the data-driven procedure from Section 4.1.

```
double expTestL2puri(NumericVector x,double a)
{
 int n = x.size();
 double n1=double(n);
 double total=0;
  for(int i=0;i<n;i++)
  {
    for(int k=0;k<n;k++)
    {
      total+=n1*n1/(a+x[i]+x[k]);
      for(int j=0;j<n;j++)
      {
        total-=2*n1/(a+x[k]+fabs(x[i]-x[j]));
        for(int l=0;l<n;l++)
        {
          total+=(1.0/(a+fabs(x[i]-x[j])+fabs(x[k]-x[l])));
        }
      }
    }
  }
 double stat=total/n1/n1/n1/n1;
 return stat;
}

bootstrapStat<-function(x,a,B=300){
  n<-length(x)
  Xs<-sample(x,B*n,replace = TRUE)
  Xs<-array(Xs,c(n,B))
  Tb<-apply(Xs/mean(Xs),2,expTestL2puri,a)
  return(Tb)
}


optimal_a<-function(y,a,B)
{
  n=length(y)
P<-rep(0,length(a))
for(k in 1:length(a)){
```

```
    ts0<-rep(0,10000)
    for(i in 1:10000){
      x<-rexp(n)
      ts0[i]<-expTestL2puri(x/mean(x),a[k])
                     ″
    C<-quantile(ts0,0.95)
    Tb<-bootstrapStat(y,a[k],B)
      P[k]<-sum(Tb>=C)/B
      ″


  m<-which.max(P)
  return(a[m])
″
```


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