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Tail risk measures using flexible parametric distributions

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Abstract

We propose a new type of risk measure for non-negative random variables that focuses on the tail of the distribution. The measure is inspired in general parametric distributions that are well-known in the statistical analysis of the size of income. We derive simple expressions for the conditional moments of these distributions, and we show that they are suitable for analysis of tail risk. The proposed method can easily be implemented in practice because it provides a simple one-step way to compute value-at-risk and tail value-at-risk. We show an illustration with currency exchange data. The data and implementation are open access for reproducibility.

MSC: 60E05, 62P05.

Keywords: Moments, multi-period risk assessment, value-at-risk

1 Introduction

Monitoring risk is one of the most difficult problems in many areas such as finance and insurance. When risk changes dynamically there is no guarantee that the distribution remains stable over time, for instance even if the same family of distributions can be assumed, there may be a drift and, moreover, dispersion may change. When the deviation from the mean is not constant over time, then we encounter the well-known concept of changing volatility.

We propose new risk measures that concentrate on the far-end tail of the distribution. We show that these new measures, under suitable mild regularity conditions, can be implemented easily because they have simple analytical (or numerical) expressions. This characteristic makes them suitable for monitoring risk, when a direct method is needed with the same protocol along time.

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A rich variety of risk measures can be calculated under the approach that is presented here, when we consider flexible distributions for non-negative random variables. Our work is inspired by the analysis of the size of income distributions, for which there is a long tradition in economics. However, our main contribution is that we find straightforward formulas for the conditional moments of the distributions. Since we concentrate on estimating the tails of the distributions and we are also concerned about being able to implement these risk measures in practice using fast and direct computation, the simple moment expressions are very convenient, for instance to compute the expectation conditional on the variable exceeding the value at risk. We believe that these new measures have a large field of application and they offer an interesting new and tractable approach for practitioners.

2 Basic result

Since we aim at analysing the tail of the distribution, our first result is about moments and, in particular, on higher order moments beyond a certain value. Our interest on moments implies that we study the expectation of the transformation of a random variable through a power function and, just like it is done in conditional tail expectation, we condition on the domain beyond a certain level.

Theorem 1 *Let X be a non-negative and continuous random variable with PDF $f(x)$, CDF $F(x)$, and we assume that $E[X^r]$ is finite for some value $r > 0$. Let us denote by $F_{(r)}$ the CDF of the rth incomplete moments, that is, $F_{(r)}(x) = \frac{\int_0^x z^r dF(z)}{E[X^r]}$, e.g. defined in Kleiber and Kotz (2003). Then, if $t > 0$ we have,*

$$E[X^r|X > t] = E[X^r] \cdot \frac{1 - F_{(r)}(t)}{1 - F(t)}. \tag{1}$$

In particular, when $t = x_\alpha$ denotes the α quantile of X, that is $\Pr(X \leq x_\alpha) = F(x_\alpha) = \alpha$, formula (1) is then,

$$E[X^r|X > x_\alpha] = E[X^r] \cdot \frac{1 - F_{(r)}(x_\alpha)}{1 - \alpha}. \tag{2}$$

Proof: The result follows directly from the definition of incomplete moments given above and standard properties of the cumulative distribution function. ■

The interest of the previous result is that conditional tail higher-order moments can be easily derived if the assumed distribution has simple expressions for the (unconditional) moments, $E(X^r)$, and for the CDF of the rth incomplete moment. As we will see below, there are some distributions for which these expressions can easily be found.

3 McDonald's model

McDonald (1984) analysed distributions for the size of income and found a comprehensive framework that allows a straightforward estimation of parameters and additional features of many distributions for non-negative random variables. The generalized gamma (GG) distribution was proposed by Stacy (1962), while the generalized beta of the first kind (GB1) and the generalized beta of the second kind (GB2), sometimes termed Generalized Beta Prime, were proposed in this context by McDonald (1984) and they are defined in terms of their probability density functions ($a,b,p,q>0$) as follows:

$$f_{GG}(x;a,p,b) = \frac{ax^{ap-1}\exp(-(x/b)^a)}{b^{ap}\Gamma(p)}, \; x>0, \tag{3}$$

$$f_{GB1}(x;a,p,q,b) = \frac{ax^{ap-1}[1-(x/b)^a]^{q-1}}{b^{ap}B(p,q)}, \; 0\le x\le b, \tag{4}$$

$$f_{GB2}(x;a,p,q,b) = \frac{ax^{ap-1}}{b^{ap}B(p,q)[1+(x/b)^a]^{p+q}}, \; x\ge 0, \tag{5}$$

and 0 otherwise. Here $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}\exp(-t)dt$ represents the gamma function and $B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt$ the beta function, where $\alpha,p,q>0$. Note that the parameter b is a scale parameter.

A random variable X with PDF (3)-(5) will be denoted by $X\sim GG(a,p,b)$, $X\sim GB1(a,p,q,b)$ and $X\sim GB2(a,p,q,b)$ respectively. These models include an important number of income distributions. As such, they have been wildly used in many applications. Here we present a few simple examples:

- The generalized gamma (**GG**) distribution includes: the exponential distribution ($a=p=1$), the classical gamma distribution ($a=1$); if $a=1$ and $p=n/2$, a chi-squared distribution with n degrees of freedom is obtained, the classical Weibull distribution ($p=1$), the half normal distribution ($a=2$ and $p=1/2$). Moreover, the two-parameter lognormal distribution is a limiting case of the generalized gamma distribution given by $a\to 0$, $p,b\to\infty$, $a^2\to\sigma^{-2}$ and $bp^{1/a}\to\mu$.
- The **GB1** distribution includes the three-parameter classical beta distribution with support $(0,b)$ if we set $a=1$ in (4). When letting $a=b=1$, we obtain the usual classical beta distribution of the first kind. Chapter 25 of the book of Johnson, Kotz, and Balakrishnan (1995) contains a careful study of beta distributions. See also Balakrishnan and Nevzorov (2004), Chapter 16.
- The **GB2** includes the usual second kind beta distribution ($a=1$), the Singh-Maddala distribution (Singh et al. (1976)) ($p=1$), the Dagum distribution (Dagum (1977)) ($q=1$), the Lomax or Pareto II distribution ($a=p=1$) and the Fisk or

log-logistic distribution ($p = q = 1$). The GB2 distribution was referred to as a Feller-Pareto distribution by Arnold (1983), including an additional location parameter.

One of the main advantages of the McDonald's family is the huge variety of particular or limiting cases that it contains. Many of the models that are basic in the analysis of size and income can be expressed in this framework. According to McDonald (1984), both of the generalized beta distributions include the generalized gamma as a limiting case.

3.1 Properties of the generalized function for the size distribution of income

In this section we describe several properties of the members of the McDonald family, which will be used in the rest of the paper. We aim at finding those characteristics that are useful to describe the tails, as we are mainly concentrated on measuring the risk.

In order to obtain the CDF of the GG distribution, we consider the incomplete gamma function ratio defined by,

$$G(x;\nu) = \frac{1}{\Gamma(\nu)} \int_0^x t^{\nu-1} \exp(-t)dt, \ \ x > 0, \tag{6}$$

with $\nu > 0$. Note that (6) corresponds to the CDF of the classical gamma distribution with shape parameter $\nu > 0$ and scale parameter $b = 1$. As a consequence,

$$x_\alpha = G^{-1}(\alpha;\nu) \tag{7}$$

represents the quantile of order α corresponding to the classical gamma distribution with shape parameter α, scale parameter $b = 1$ and PDF $f(x) = \frac{x^{\nu-1}e^{-x}}{\Gamma(\nu)}$.

Using (6), the CDF of (3) is given by,

$$F_{GG}(x;a,p,b) = G((x/b)^a;p), \ \ x \geq 0. \tag{8}$$

Now, we consider the incomplete beta function ratio defined by,

$$B(x;p,q) = \frac{1}{B(p,q)} \int_0^x t^{p-1}(1-t)^{q-1}dt, \ \ 0 \leq x \leq 1 \tag{9}$$

with $p,q > 0$. Function (9) corresponds to the CDF of the classical beta distribution with PDF $f(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}$. Therefore,

$$x_\alpha = B^{-1}(\alpha;p,q) \tag{10}$$

represents the quantile of order α of a classical beta distribution with parameter (p,q).

The CDF of the GB1 distribution is:

$$F_{GB1}(x;a,p,q,b) = B((x/b)^a;p,q), \;\; 0 \leq x \leq b, \tag{11}$$

where $B(\cdot;\cdot,\cdot)$ is defined in (9).

The CDF of the GB2 can be easily defined in terms of the incomplete beta function ratio (9) and their CDF is given by,

$$F_{GB2}(x;a,p,q,b) = B\left(\frac{(x/b)^a}{1+(x/b)^a};p,q\right), \;\; x \geq 0. \tag{12}$$

Butler and McDonald (1989) showed that many inequality measures depend upon the incomplete moments of the income distribution, see also Kleiber and Kotz (2003). They showed that they are easily calculated for a very broad family of distributions because they possess a closure property. This is the case of the GG, GB1 and the GB2 distributions. The CDF, the distribution of the rth incomplete moment $X_{(r)}$ and the moments of the GG, GB1 and GB2 distributions are summarized in Table 1.

Table 1: *The CDF, the distribution of the rth incomplete moment $X_{(r)}$ and the moments of the GG, GB1 and GB2 distributions. For the GB2 distribution $E[X^r]$ and $X_{(r)}$ exist if $q < r/a$.*

Distribution	GG	GB1	GB2
CDF	$G((x/b)^a;p)$	$B((x/b)^a;p,q)$	$B\left(\frac{(x/b)^a}{1+(x/b)^a};p,q\right)$
$X_{(r)}$	$GG(a,p+\frac{r}{a},b)$	$GB1(a,p+\frac{r}{a},q,b)$	$GB2(a,p+\frac{r}{a},q-\frac{r}{a},b)$
$E[X^r]$	$\frac{b^r\Gamma(p+\frac{r}{a})}{\Gamma(p)}$	$\frac{b^rB(p+\frac{r}{a},q)}{B(p,q)}$	$\frac{b^rB(p+\frac{r}{a},q-\frac{r}{a})}{B(p,q)}$

Summarized from Butler and McDonald (1989) and Kleiber and Kotz (2003)

3.2 Estimation of the GG, GB1 and GB2

In order to implement the calculation of the tail risk measures for the distributions of the McDonald family, it is necessary to provide a simple way to fit these distributions. These models can be estimated by maximum likelihood but, as already noted by Prentice (1974) among others, maximization can be difficult. Alternatively, moment estimates can be used.

For a given data set, the sample moments should be calculated and then the parameter estimates can be found, solving the expressions for the theoretical moments given in the last row of Table 1. All positive moments exist for the GG and the GB1. It is not the case for the GB2. Estimation by the method of moments up to four implies the existence of moments up to four in the GB2 case, which implies constraints of the parameters space.

3.3 Conditional moments

Using the results of the previous sections, we can obtain simple expressions for the tail moments. These results follow immediately.

3.3.1 Formulation for the GG Distribution

For the GG distribution, the conditional moments in formula (2) can be expressed as,

$$E[X^r|X>x_\alpha] = \frac{b^r\Gamma(p+r/a)}{(1-\alpha)\Gamma(p)} \cdot \left\{1 - G\left(\left(\frac{x_\alpha}{b}\right)^a ; p+\frac{r}{a}\right)\right\}, \tag{13}$$

where the quantile, also called value at risk (VaR) is

$$x_\alpha = b \cdot \left\{G^{-1}(\alpha;p)\right\}^{1/a}.$$

3.3.2 Formulation for the GB1 Distribution

For the GB1 distribution, the conditional moments in formula (2) are expressed as follows:

$$\begin{aligned} E[X^r|X>x_\alpha] &= \frac{b^r B(p+r/a,q)}{(1-\alpha)B(p,q)} \cdot \left\{1 - B\left(\left(\frac{x_\alpha}{b}\right)^a ; p+\frac{r}{a},q\right)\right\}, \\ &= \frac{b^r\Gamma(p+r/a)\Gamma(p+q)}{(1-\alpha)\Gamma(p+q+r/a)\Gamma(p)} \cdot \left\{1 - B\left(\left(\frac{x_\alpha}{b}\right)^a ; p+\frac{r}{a},q\right)\right\}, \end{aligned} \tag{14}$$

where

$$x_\alpha = b \cdot \left\{B^{-1}(\alpha;p,q)\right\}^{1/a}.$$

3.3.3 Formulation for the GB2 Distribution

For the GB2 distribution, formula (2) gives the following expression for the conditional moments,

$$\begin{aligned} E[X^r|X>x_\alpha] &= \frac{b^r B(p+r/a,q-r/a)}{(1-\alpha)B(p,q)} \cdot \left\{1 - B\left(\frac{(x_\alpha/b)^a}{1+(x_\alpha/b)^a} ; p+\frac{r}{a},q-\frac{r}{a}\right)\right\}, \\ &= \frac{b^r\Gamma(p+r/a)\Gamma(q-r/a)}{(1-\alpha)\Gamma(p)\Gamma(q)} \cdot \left\{1 - B\left(\frac{(x_\alpha/b)^a}{1+(x_\alpha/b)^a} ; p+\frac{r}{a},q-\frac{r}{a}\right)\right\}, \end{aligned} \tag{15}$$

if $q > r/a$ where

$$x_\alpha = b \cdot \left\{\frac{B^{-1}(\alpha;p,q)}{1-B^{-1}(\alpha;p,q)}\right\}^{1/a}.$$

4 Tail risk measures

One of the advantages of having obtained the expressions in the previous section is that it is straightforward to define tail risk measures. This means that we concentrate on the part of the distribution that exceeds a certain level, for instance a certain quantile. In fact, the expected shortfall is one of the easiest forms of tail risk measure, because in plain words, it measures the expected loss beyond a given quantile level and, as such, is only concerned about the size of losses in the worst-case part of the domain.

The different risk measures are given by,

$$E[X|X>x_\alpha] = m, \tag{16}$$

$$var[X|X>x_\alpha] = E[(X-m)^2|X>x_\alpha], \tag{17}$$

$$\gamma_1[X|X>x_\alpha] = \frac{E[(X-m)^3|X>x_\alpha]}{\{var[X|X>x_\alpha]\}^{3/2}}, \tag{18}$$

$$\gamma_2[X|X>x_\alpha] = \frac{E[(X-m)^4|X>x_\alpha]}{\{var[X|X>x_\alpha]\}^{2}} - 3. \tag{19}$$

These tail risk measures can be written in terms of the tail moments

$$m_r = E[X^r|X>x_\alpha], \;\; r = 1,2,\ldots$$

as ($m_1 = m$),

$$var[X|X>x_\alpha] = m_2 - m^2, \tag{20}$$

$$\gamma_1[X|X>x_\alpha] = \frac{m_3 - 3\cdot m\cdot m_2 + 2\cdot m^3}{\{m_2-m^2\}^{3/2}}, \tag{21}$$

$$\gamma_2[X|X>x_\alpha] = \frac{m_4 - 4\cdot m\cdot m_3 + 6\cdot m^2\cdot m_2 - 3\cdot m^4}{\{m_2-m^2\}^{2}} - 3. \tag{22}$$

Note that the notion of tail value at risk (TVaR) corresponds to m_1. Risk measures other than the value at risk and the tail value at risk, such as GlueVaR proposed by Belles-Sampera, Guillén, and Santolino (2014) can also be calculated. Guillen, Prieto, and Sarabia (2011) analysed risk measures in tails that have a Pareto shape and Generalized beta-generated distributions were studied in Alexander et al. (2012).

5 Case study: tail measures in currency exchange series

Series of daily currency exchange are considered. An example using data from currency exchanges is suitable because exchanges always take a positive value. Three currency exchanges were selected: Australian to US dollars, US dollar to British pound sterling and US dollar to Yen. We only show here the results for the US dollar to British pound sterling with a series ranging from January 1971 to July 2014. We have selected this particular time frame because it corresponds to a long interval covering several periods of crisis, and thus serves as a good illustration. The other exchange rates lead to similar conclusions, with the exception of the location in time of the periods of high risk, which do not necessarily coincide with those between US dollar and British pound. Results for the other currencies together with the R implementation can be obtained from the authors. Figure 1 displays the raw data for daily exchange rate series and Table 2 presents some summary statistics.

Table 2: *Descriptive summary of the observed exchange rate between the US dollar and the British pound from 1970 to 2014.*

Observed USD/GBP exchange	(N = 10921)
min	1.052
max	2.644
median	(IQR) 1.67 (1.56, 1.91)
mean	(95% CI) 1.77 (1.76, 1.77)
second moment	3.22
third moment	6.05
fourth moment	11.78

The second, third and fourth moments for the whole observed period (from 1970 to 2014) do not necessarily reflect the relative size with respect to the first moment at every window.

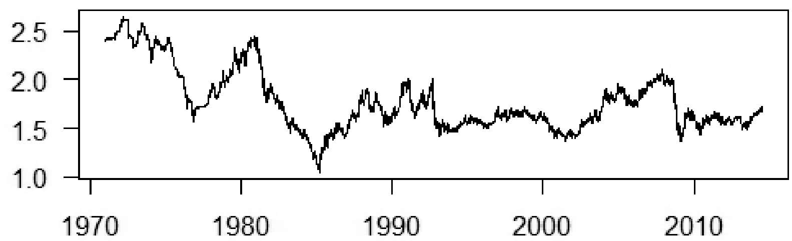

Figure 1: *US dollar/British pound exchange rate from 1971 to 2014.*

The fourth moment is much larger than the first, even if exchange rates means are smaller than 2, which implies that the importance of the fourth moment in the minimization procedure is the largest. A weighted method of moments that gives roughly the same order of magnitude to all four moments could be compared with the unweighted method. In the same vein, more recent observations could be weighted more that distant past observations in a rolling window. There are many possibilities on how to construct such weights and there is not a consensus in finance about this. We have preferred to leave this point as an open question for further research.

A rolling window is implemented, so that the tail risk is calculated using a window of 250 observations. Each new window drops the first observation and adds a new one at the end of the 250 observation days. In this way, a long daily series of tail risk measures can be obtained, using in each case a window of 250 days.

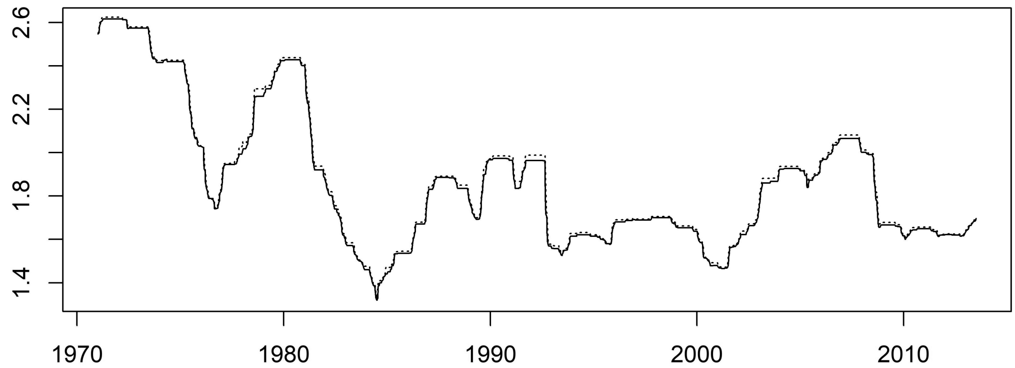

Figure 2: *Empirical estimates of 95% value at risk (solid line) and tail value at risk (dashed line) for the exchange rate between the US dollar and the British pound from 1971 to 2014, in 250-days windows.*

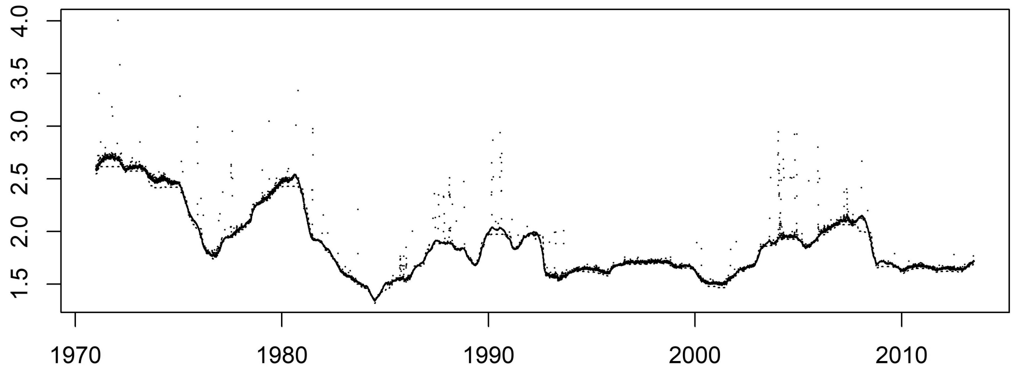

Figure 3: *McDonald model (GG) estimates of 95% value at risk for exchange rates between the US dollar and British pound from 1971 to 2014, in 250-days windows (dot points). Empirical estimates are presented in dashed lines.*

Our optimization method is based on minimizing the Euclidean distance between the theoretical moments and the empirical moments, where we always checked that distance was close enough to zero, less than 0.001. We have also compared parametric estimates versus empirical estimates as suggested by McDonald and Ransom (1979) . We always achieved convergence in our examples. However, as suggested by one of the reviewers, a useful recommendation when implementing this kind of optimization in a rolling window is to take the result of parameter estimation (in the previous window) as the seed in numerical optimization in the following one.

Figure 2 shows the results and compares the tail analysis for a 95% value at risk (VaR) and the 95% tail value at risk when using an Empirical CDF. Figures 3 and 4 present the analysis of the 95% VaR and the 95% TVaR of the McDonald generalized gamma model for the exchange rate, respectively. To save space, we only present the graphical results for the GG distribution but the results (available upon request) are similar when we use de GB1 and GB2 distributions. The conclusion is that the proposed model is able to capture fluctuations of the risk in the exchange rate that the empirical analysis cannot capture. Note that the spikes in specially risky days are spotted much better with our method. There are periods of high risk around 1973 (oil crisis), 1985 (international intervention in the currency markets to depreciate the dollar), 1987 (market crash), 2004 (dot-com bubble) and 2008 (Lehman Brothers and global financial crisis). If the empirical conditional distribution function was used, the tail risk would have been substantially underestimated. The McDonald approach seems to provide values that are larger than those provided by the empirical approach and they seem to be much more sensitive to daily updates in the rolling window.

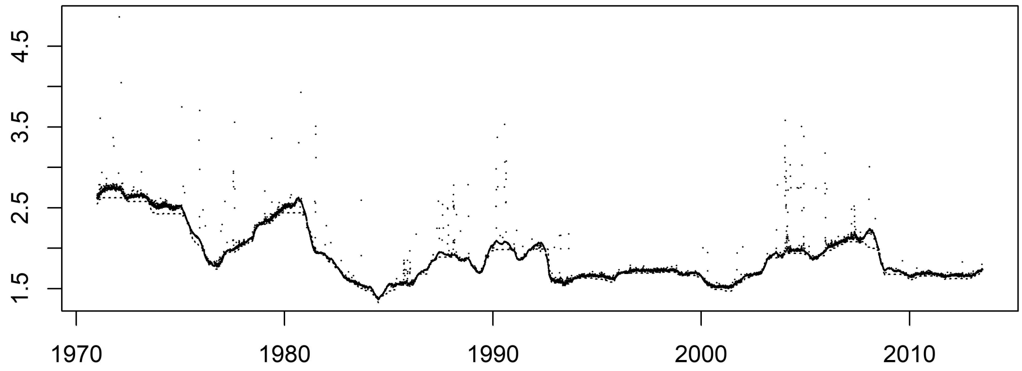

Figure 4: *McDonald model (GG) estimates of 95% tail value at risk for exchange rates between the US dollar and the British pound from 1971 to 2014, in 250-days windows (dot points). Empirical estimates are presented in dashed lines.*

Table 3 presents some summary statistics of the empirical and the estimated McDonald (GG, GB1 and GB2) 95% value at risk and tail value at risk in 250-days windows from 1971 to 2014 of the exchange rate between the US dollar and the British pound. As expected after inspection of Figures 3 and 4, the summary statistics of the value at risk and the tail value at risk are higher when using the GG distribution than when using the empirical CDF.

Figures 3 and 4 offer the comparative analysis for value at risk and tail value at risk to see the parametric estimates versus the empirical. The inferior stability of parametric estimates could indeed speak against the parametric method, but it could also show that fitting a parametric distribution requires to look at the whole domain, making inference about the tail more dependent on the location and shape than empirical risk measures. Empirical estimates of the quantiles differ from estimates based on the parametric fit because they only sort observations and choose the value (or an interpolation of two values) that corresponds to the chosen confidence level, here 95%. If there is a large extreme suddenly appearing on the right tail, the quantile may not react to that phenomenon. This is the main disadvantage of working with quantiles. When looking at the empirical tail conditional expectation estimates (dashed line in Figure 2), there is only a slight increase of the tail value at risk compared to the value at risk. In contrast, parametric distributions are fitted with all the information in the data.

Table 3: *Summary of the empirical and the estimated McDonald (GG, GB1 and GB2) 95% value at risk and tail value at risk in 250-days windows from 1971 to 2014 of the exchange rate between the US dollar and the British pound.*

	Daily exchange rate USD/GBP (N = 10,671)			
		McDonald models		
	Empirical	GG	GB1	GB2
Value at Risk				
min	1.32	1.34	1.32	1.35
max	2.62	4.00	2.87	3.02
median	1.80	1.80	1.78	1.88
(IQR)	(1.62, 1.99)	(1.65, 2.03)	(1.63, 1.99)	(1.65, 2.08)
mean	1.87	1.89	1.86	1.92
(95% CI)	(1.86, 1.87)	(1.88, 1.90)	(1.85, 1.86)	(1.91, 1.92)
Tail Value at Risk				
min	1.33	1.36	1.23	1.39
max	2.63	4.86	2.96	3.29
median	1.82	1.83	1.79	1.93
(IQR)	(1.64, 2.00)	(1.67, 2.07)	(1.64, 2.01)	(1.68, 2.16)
mean	1.88	1.92	1.87	1.98
(95% CI)	(1.87, 1.88)	(1.91, 1.93)	(1.86, 1.87)	(1.97, 1.98)

In our case study, daily observations correspond to a different random variable, for which we only have exactly one observation. When we deploy a rolling window, our hypothesis is that the distribution remains stable during that window period and that observations are independent. Empirical risk estimates of value at risk and tail value at risk have been extensively used in the literature, knowing that they are very robust. But when analysing risk, and in our approach, we prefer a parametric approach that considers the size of all the observations.

In order to take into consideration sample size issues, we have tried wide windows observations of 500 and 750 daily data. The conclusions did not change. As noted by one of the reviewers, time series characteristics may indeed be interfering in the estimation. Standard errors may be affected by the existence of positive and significant correlation between subsequent daily observations, but our application does not address inference questions.

6 Conclusions

We conclude that the McDonald model is a suitable framework to analyse tail risk and we show that it can easily be implemented with moment estimates, it is fast and it does not require considerable computational effort. The main importance of our proposed approach is about the implementation.

Even if expressions for incomplete moments of income distributions had been analysed before, the focus there was on their link to inequality measures. Our added value here is about the analysis of conditional moments and their relationship with risk measures such as the tail conditional expectation. These exact expressions had not been implemented before. By finding the link between moments, incomplete moments of income distributions and risk measures we facilitate the task of risk analysts.

Tail risk analysis can be done as fast as when the empirical distribution is assumed, because parameter can be fitted using the first moments. Then, tail risk is computed immediately from the expressions presented above.

Butler and McDonald (1989) mentioned that in many fields of applications the entire shape of the distribution, not just its mean, is important and they gave an empirical example where they calculated normalized incomplete moments or moment distributions of the GB1, GB2 and GG in US income data for a series of years. They used maximum likelihood estimation on grouped data. They concluded that these income distribution moments characterize important properties of interest in an analysis of the distribution of economic data (see also Butler and McDonald (1987)). Our practical contribution concentrates on the tail. We provide a moment estimation procedure that is fast in practice, produces a quick answer (through the remark given by Theorem 1) and improves the results of empirical measures.

The proposed methodology is useful in the analysis of financial time series, since it has the capability to detect periods where the risk is high and the results are realistic in the most of cases. However, isolated points can suggest non-stability on parameter estimation.

We have not addressed the question of the relative merits of alternative estimation techniques in this paper. McDonald and Ransom (1979) noted that the techniques of maximum likelihood estimation and method of moments are not directly appropriate for the case in which grouped data is used. As a practical tool, these authors suggest to check the agreement between the implied, i.e. substituting the parameter estimates in the expression for the mathematical expectation, and empirical estimates of the mean. Their main concern is about the fact that they are using grouped rather than individual data. Since we are working on individual observations we believe that both maximum likelihood estimation and the method of moments estimation are suitable. However, when fitting a GG distribution, Prentice (1974) and earlier authors note that maximization of the likelihood function with Newton-Raphson method does not work well and that the existence of solutions to the log-likelihood equations is sometimes in doubt. For the GG distribution, the `flexsurv` R package (Jackson, 2016) could be used.

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References

Alexander, C., G. Cordeiro, E. Ortega, and J. Sarabia (2012). Generalized beta-generated distributions. *Computational Statistics and Data Analysis*, 56, 1880–1897.

Arnold, B. (1983). *Pareto Distributions*. International Co-operative Publishing House.

Balakrishnan, N. and V. Nevzorov (2004). *A Primer on Statistical Distributions*. John Wiley & Sons.

Belles-Sampera, J., M. Guillén, and M. Santolino (2014). Beyond value-at-risk: Gluevar distortion risk measures. *Risk Analysis*, 34, 121–134.

Butler, R. and J. McDonald (1987). Interdistributional income inequality. *Journal of Business and Economic Statistics*, 5, 13–18.

Butler, R. and J. McDonald (1989). Using incomplete moments to measure inequality. *Journal of Econometrics*, 42, 109–119.

Dagum, C. (1977). New model of personal income-distribution-specification and estimation. *Economie appliquée*, 30, 413–437.

Guillen, M., F. Prieto, and J. Sarabia (2011). Modelling losses and locating the tail with the pareto positive stable distribution. *Insurance: Mathematics and Economics*, 49, 454–461.

Jackson, C. (2016). flexsurv: a platform for parametric survival modeling in R. *Journal of Statistical Software*, 70.

Johnson, N., S. Kotz, and N. Balakrishnan (1995). Continuous univariate distributions, vol. 2 of wiley series in probability and mathematical statistics: applied probability and statistics.

Kleiber, C. and S. Kotz (2003). *Statistical Size Distributions in Economics and Actuarial Sciences*, Volume 470. John Wiley & Sons.

McDonald, J. (1984). Some generalized functions for the size distribution of income. *Econometrica: Journal of the Econometric Society*, 647–663.

McDonald, J. and M. Ransom (1979). Functional forms, estimation techniques and the distribution of income. *Econometrica: Journal of the Econometric Society*, 1513–1525.

Prentice, R. (1974). A log gamma model and its maximum likelihood estimation. *Biometrika*, 61, 539–544.

Singh, S., G. Maddala, et al. (1976). A function for size distribution of incomes. *Econometrica*, 44, 963–970.

Stacy, E. (1962). A generalization of the gamma distribution. *The Annals of Mathematical Statistics*, 33, 1187–1192.