SORT 49 (1) January-June 2025, 43-72 DOI: 10.57645/20.8080.02.22

On statistical model extensions based on randomly stopped extremes

Jordi Valero and Josep Ginebra[1]

Abstract

The maxima and the minima of a randomly stopped sample of a random variable, X, together with two newly defined random variables that make X into the maxima or minima of a randomly stopped sample of them, can be used to define statistical model transformation mechanisms. These transformations can be used to define models for extreme-value data that are not grounded on large sample theory. The relationship between the stopping model and characteristics of the corresponding model transformations obtained is investigated. In particular, one looks into which stopping models make these model transformations into model extensions, and which stopping models lead to statistically stable extensions in the sense that using the model extension a second time leaves the extended model unchanged. The stopping models under which the extensions based on randomly stopped maxima and their inverses coincide with the extensions based on randomly stopped minima and their inverses are also characterized. The advantages of using models obtained through these model extension mechanisms instead of resorting to extreme-value models grounded on asymptotic arguments is illustrated by way of examples.

MSC: *62E10, 62A99, 60E10.*

Keywords: *Marshall-Olkin extension, extreme value, randomly stopped maximum, randomly stopped minimum, statistical stability, stopping model.*

1. Introduction

In disciplines such as hydrology, meteorology, ecology, seismology, actuarial sciences, civil engineering or finance, there is a need for statistical models to analyze extreme-valued data, like the largest single-event rainfall or the magnitude of the strongest earth-

[1] Address for correspondence: Department of Statistics and O.R., Polytechnic University of Catalonia, Avgda. Diagonal 647, 6ª Planta, 08028 Barcelona, Spain (E-mail: josep.ginebra@upc.es)
Received: September 2023.
Accepted: November 2024.

quake in a year. In these settings researchers most often resort to the use of the generalized extreme-value model, which is grounded on large sample theory that only applies as an approximation when sample sizes are large enough.

Hence, there is a need for statistical models for extreme-valued data that can be grounded on finite-sample theory. One framework that provides that ground, models the number of events in a year, like the number of rainfalls or of earthquakes, through a random variable, N, with a given stopping model, it models the magnitude of the events in that year as a sample of i.i.d. observations, $(X_1, \ldots, X_N)$, with a given stopped model, and it assumes that one observes the maxima or the minima, Y, of that randomly stopped sample. Models defined like the one for Y are also useful in reliability, where the minimum (or maximum) of a randomly stopped sample from a lifetime distribution serves as a model for the lifetime of a series (or parallel) system.

Marshall and Olkin (1997) obtained statistical models of this kind by extending an initial statistical model through the distribution of the minimum and of the maximum of a geometrically stopped sample of independent observations with a distribution in the initial family. This statistical model transformation mechanism has proved extremely fruitful in practice, as the more than seventeen hundred citations of that paper indicate.

One nice feature of model transformations based on geometrically stopped extremes is that they always work as model extensions, because the initial family of distributions is always included in the new family. A second interesting feature of these geometrically stopped extreme extensions is that they are statistically stable in the sense that the extended model can not be further extended by using that same extension mechanism a second time. These two features are not in place in general, when transforming statistical models through randomly stopped maxima and minima with a stopping model different from geometric. In fact, Marshall and Olkin (1997) conjectures that this kind of stability can only be obtained through geometrically stopped extremes.

Here these issues are investigated in full generality, by looking into all model transformations defined through the maxima or the minima of N-stopped random samples of X, for any given stopping model for N and any given stopped model for X.

On top of looking into randomly stopped extreme model extensions beyond geometric stopping, we also propose two new model transformation mechanisms based on two new random variables defined to be the ones that make X into the randomly stopped maxima and the randomly stopped minima of them, which we label as the N-maxprecursor and the N-minprecursor of X. These transformations can be viewed as the inverse transformations of N-stopped maxima and of N-stopped minima of X, and the statistical models obtained through them can be used to learn about the magnitude of events, X, based on their frequency N and their extreme values Y.

Finally, on top of these four basic model transformation mechanisms based on randomly stopped maxima and minima and on their inverses, we also propose another two new pair of transformation mechanisms that combine N-stopped maxima of X with their inverses, and combine N-stopped minima of X with their inverses. Under geometric stopping, these combined model transformation mechanisms coincide with the

Marshall-Olkin extension mechanism, and they work as model extensions under any stopping model, which is why we consider them to be the natural way to generalize Marshall-Olkin when the stopping model is not geometric.

The relationship between characteristics of the stopping model and characteristics of all the corresponding model transformations considered is studied. The first objective is to identify which stopping models lead to transformations that always work as model extensions, and the second objective is to identify which stopping models lead to model extensions that are statistically stable, in the sense that they do not further extend the initial model beyond the first use.

The second objective leads to the investigation of the class of stopping models that are closed under probability generating function (pgf) composition, because that is a necessary condition for the corresponding randomly stopped extreme extensions to be stable. This investigation helps us to disprove by way of examples the conjecture that only geometric stopping models lead to statistically stable extensions.

The paper also looks into the reversibility conditions required of stopping models so that the model extensions built based on N-stopped maxima and their inverses coincide with the model extensions built based on N-stopped minima and their inverses, which is a property satisfied in particular by the extensions based on geometric stopping.

The paper illustrates through examples the advantages of modeling extreme-valued data with models obtained through randomly stopped extreme extensions instead of resorting to the usual generalized extreme-value model backed through large sample arguments. We also use examples to help understand the rationale behind the use of the models obtained through the new model extension mechanisms that use the inverse of N-stopped maxima or minima.

The paper is organized as follows. Section 2 defines randomly stopped extreme and extreme-precursor random variables, and presents the four basic and four combined model transformation mechanisms that will be investigated, and Section 3 illustrates the use of models obtained with these transformations to deal with extreme-value data. Section 4 introduces the definition of statistically stable model transformation. Section 5 defines extreme-reversible and auto-reversible stopping models, and Section 6 looks into stopping models that are closed under pgf composition, which are the ones that yield statistically stable transformations. Section 7 relates these and other properties of the stopping model with features of the corresponding model transformations, and Section 8 presents examples of statistically stable randomly stopped extreme extensions.

2. Statistical models based on randomly stopped extremes

2.1. Randomly stopped extremes and extreme precursors

Let X be a real valued random variable defined through its cumulative distribution function, F_X, and let N be a positive integer valued random variable, with $\Pr(N = 0) = 0$, defined through its probability generating function (pgf), h_N. Assume that one observes

n independent copies of X, X_i, where n is a realization of the stopping variable N independent of the X_i.

The *N-stopped maximum of X*, which we denote by $\max_N(X)$, is the random variable $Y = \max(X_1, \cdots X_N)$ with cumulative distribution function:

$$F_{\max_N(X)} = h_N(F_X),$$

and the *N-stopped minimum of X*, which we denote by $\min_N(X)$, is the random variable $Y = \min(X_1, \cdots, X_N)$ with survival function $h_N(S_X)$, where $S_X = 1 - F_X$, and therefore with cdf:

$$F_{\min_N(X)} = 1 - h_N(1 - F_X) = \overline{h}_N(F_X),$$

where $\overline{h}_N(t) = 1 - h_N(1-t)$, which will be denoted as the *conjugate function* of $h_N(t)$.

These two random variables are studied for example in Raghundanan and Patil (1972), Shaked (1975), Consul (1984), Gupta and Gupta (1984), Rohatgi (1987), Shaked and Wong (1997), in pp.155-157 of Arnold, Balakrishnan and Nagaraja (1992) and in Louzada, Beret and Franco (2012).

Next, two new random variables that play a central role in what follows are introduced. They arise from the fact that given any N and any X, one can always interpret X to be the N-stopped maximum and the N-stopped minimum of the two random variables defined next.

Definition 1. *Given any stopping variable N and any real valued random variable X as defined above, let the N-maxprecursor of X, denoted as $\max_N^{-1}(X)$, be the random variable Y with cdf:*

$$F_{\max_N^{-1}(X)} = h_N^{-1}(F_X),$$

and let the N-minprecursor of X, denoted as $\min_N^{-1}(X)$, be the random variable Y with cdf:

$$F_{\min_N^{-1}(X)} = \overline{h}_N^{-1}(F_X).$$

The properties of h_N and of $\overline{h}_N$ presented in Section 5.1 guarantee that they are always invertible and therefore that $F_{\max_N^{-1}(X)}$ and $F_{\min_N^{-1}(X)}$ are always properly defined cdf's. As a consequence, the random variables $\max_N^{-1}(X)$ and $\min_N^{-1}(X)$ will exist for any N and any X.

By definition, X is always the N-stopped maximum of $\max_N^{-1}(X)$, the N-stopped minimum of $\min_N^{-1}(X)$, the N-maxprecursor of $\max_N(X)$, and the N-minprecursor of $\min_N(X)$,

$$X = \max_N(\max_N^{-1}(X)) = \max_N^{-1}(\max_N(X)) = \min_N(\min_N^{-1}(X)) = \min_N^{-1}(\min_N(X)),$$

which is why we denote N-maxprecursors and minprecursors as the inverses of the N-stopped maxima and minima.

2.2. Statistical model transformations based on randomly stopped extremes

Let the family of distributions $\mathcal{X} = \{X_\theta : F_{X_\theta},\ \theta \in \Theta\}$ be a statistical model defined on $x \in S \subseteq \mathbb{R}$, with parameter space Θ, where F_{X_θ} is the cdf of X_θ.

Let $\mathcal{N} = \{N_\delta : h_{N_\delta} = \sum_{n=1}^{\infty} p_n(\delta)t^n,\ \delta \in \mathcal{D}\}$ be a statistical model defined on the positive integers, $n \in \mathbb{N}^+$, with parameter space $\mathcal{D}$, where $p_n(\delta) = \Pr(N_\delta = n)$ and where h_{N_δ} is the pgf of N_δ. We denote $\mathcal{N}$ as the stopping model.

Note that by definition in this paper it will always be assumed that stopping models, $\mathcal{N}$, are always such that $\Pr(N_\delta = 0) = 0$ for any $\delta \in \mathcal{D}$.

We next define four basic mechanisms,$\mathcal{T}(\cdot)$, that transform the initial statistical model, $\mathcal{X}$, into a new statistical model, $\mathcal{Y} = \mathcal{T}(\mathcal{X})$, through the N-stopped maximum (minimum) of $X \in \mathcal{X}$, and through the N-maxprecursors (N-minprecursors) of $X \in \mathcal{X}$, with $N \in \mathcal{N}$.

Definition 2. *Given any statistical model $\mathcal{X}$ and any stopping model $\mathcal{N}$ as defined above, let $\max_{\mathcal{N}}(\mathcal{X})$ and $\max^{-1}_{\mathcal{N}}(\mathcal{X})$ denote the statistical models defined as:*

$$\max_{\mathcal{N}}(\mathcal{X}) = \{Y_{\theta,\delta} : F_{Y_{\theta,\delta}} = h_{N_\delta}(F_{X_\theta}),\ \theta \in \Theta, \delta \in \mathcal{D}\},$$

$$\max^{-1}_{\mathcal{N}}(\mathcal{X}) = \{Y_{\theta,\delta} : F_{Y_{\theta,\delta}} = h^{-1}_{N_\delta}(F_{X_\theta}),\ \theta \in \Theta, \delta \in \mathcal{D}\}.$$

Likewise, let $\min_{\mathcal{N}}(\mathcal{X})$ and $\min^{-1}_{\mathcal{N}}(\mathcal{X})$ denote the statistical models defined as:

$$\min_{\mathcal{N}}(\mathcal{X}) = \{Y_{\theta,\delta} : F_{Y_{\theta,\delta}} = \overline{h}_{N_\delta}(F_{X_\theta}),\ \theta \in \Theta, \delta \in \mathcal{D}\},$$

$$\min^{-1}_{\mathcal{N}}(\mathcal{X}) = \{Y_{\theta,\delta} : F_{Y_{\theta,\delta}} = \overline{h}^{-1}_{N_\delta}(F_{X_\theta}),\ \theta \in \Theta, \delta \in \mathcal{D}\}.$$

These two pairs of basic transformations do not always work as model extensions. Instead, the two pairs of combined transformations defined next work as model extensions for any $\mathcal{X}$, even when one of the two new parameters is fixed. They are the family of all N-stopped maximum (minimum) of all N-maxprecursors (N-minprecursors) of X, and viceversa.

Definition 3. *Given any statistical model $\mathcal{X}$ and any stopping model $\mathcal{N}$ as defined above, let $\max_{\mathcal{N}}(\max^{-1}_{\mathcal{N}}(\mathcal{X}))$ and $\max^{-1}_{\mathcal{N}}(\max_{\mathcal{N}}(\mathcal{X}))$ denote the statistical models defined as:*

$$\max_{\mathcal{N}}(\max^{-1}_{\mathcal{N}}(\mathcal{X})) = \{Y_{\theta,\delta_1,\delta_2} : F_{Y_{\theta,\delta_1,\delta_2}} = h_{N_{\delta_2}} \circ h^{-1}_{N_{\delta_1}}(F_{X_\theta}),\ \theta \in \Theta, \delta_1, \delta_2 \in \mathcal{D}\},$$

$$\max^{-1}_{\mathcal{N}}(\max_{\mathcal{N}}(\mathcal{X})) = \{Y_{\theta,\delta_1,\delta_2} : F_{Y_{\theta,\delta_1,\delta_2}} = h^{-1}_{N_{\delta_2}} \circ h_{N_{\delta_1}}(F_{X_\theta}),\ \theta \in \Theta, \delta_1, \delta_2 \in \mathcal{D}\}.$$

Likewise, let $\min_{\mathcal{N}}(\min^{-1}_{\mathcal{N}}(\mathcal{X}))$ and $\min^{-1}_{\mathcal{N}}(\min_{\mathcal{N}}(\mathcal{X}))$ denote the statistical models:

$$\min_{\mathcal{N}}(\min^{-1}_{\mathcal{N}}(\mathcal{X})) = \{Y_{\theta,\delta_1,\delta_2} : F_{Y_{\theta,\delta_1,\delta_2}} = \overline{h}_{N_{\delta_2}} \circ \overline{h}^{-1}_{N_{\delta_1}}(F_{X_\theta}),\ \theta \in \Theta, \delta_1, \delta_2 \in \mathcal{D}\},$$

$$\min^{-1}_{\mathcal{N}}(\min_{\mathcal{N}}(\mathcal{X})) = \{Y_{\theta,\delta_1,\delta_2} : F_{Y_{\theta,\delta_1,\delta_2}} = \overline{h}^{-1}_{N_{\delta_2}} \circ \overline{h}_{N_{\delta_1}}(F_{X_\theta}),\ \theta \in \Theta, \delta_1, \delta_2 \in \mathcal{D}\}.$$

Note that these statistical model transformation mechanisms can also be used to generate statistical models, $\mathcal{Y}$, starting from a single initial random variable, $\mathcal{Y} = \mathcal{T}(X)$.

Using our notation, the Marshall-Olkin extension of $\mathcal{X}$ is defined to be $\max_{\mathcal{N}}(\mathcal{X}) \cup \min_{\mathcal{N}}(\mathcal{X})$ when $\mathcal{N}$ is the geometric stopping model. In Sections 7.2 and 7.4 it will be argued that for stopping models other than geometric the transformation $\max_{\mathcal{N}}(\mathcal{X}) \cup \min_{\mathcal{N}}(\mathcal{X})$ does not always work as an extension, but that under geometric stopping this transformation coincides with the four model transformations in Definition 3, which do work as extensions under any stopping model. As a consequence, we will propose Definition 3 and not $\max_{\mathcal{N}}(\cdot) \cup \min_{\mathcal{N}}(\cdot)$ to be the natural way to generalize Marshall-Olkin when using stopping models different from geometric.

3. Examples of the use of randomly stopped extreme models

The examples presented here illustrate the advantage in using models defined through the randomly stopped extreme transformations in Definition 2 instead of using the generalized extreme-value model, and they help understand the practical relevance of the randomly stopped extreme-precursor models also considered in that definition. The examples also touch on the rationale behind the use of the model extensions proposed in Definition 3.

3.1. On the usefulness of randomly stopped extreme models

Let's assume for example that one has data on the rainfall in the largest rain event of a year, Y_i, for a set of m years, $(y_1, \ldots, y_m)$. This kind of data is usually modeled through the three parameter generalized extreme-value model, because it is the limiting model for properly normalized extreme values when the rainfall in an event is i.i.d., and the number of rainfall events in a year grows.

As an alternative way to model this kind of data one can assume that the number of rain events in the i-th year, N_i, is random and can be modeled through a specific stopping model, $\mathcal{N}$, and that the rainfall in the set of N_i events is a sample of i.i.d. observations, $(X_1, \ldots, X_{N_i})$, from a specific model, $\mathcal{X}$. In this framework the statistical model for the largest rainfall in the i-th year, $Y_i = \max(X_1, \ldots, X_{N_i})$, is the $\mathcal{Y} = \max_{\mathcal{N}}(\mathcal{X})$ considered in Definition 2 for that $\mathcal{N}$ and that $\mathcal{X}$.

In particular, for simplicity here it will be assumed that the stopping model for the number of rain events, N_p, is the Logarithmic(p) model covered in Example 5.2 and in Appendix 1, and that the model for the rainfall in an event, X_λ, is the Exponential(λ) with cdf $F_{X_\lambda}(x) = 1 - e^{-\lambda x}$. In that case, the model for the largest rainfall of the year, $(Y_1, \ldots, Y_m)$, is the logarithmic stopped maximum of an exponential,

$$\mathcal{Y}_{Lg-Exp} = \{Y_{p,\lambda} : F_{Y_{p,\lambda}} = h_{N_p}(F_{X_\lambda}) = \frac{\log(1 - p + pe^{-\lambda y})}{\log(1-p)}, \lambda \in (0,\infty), p \in (0,1)\}.$$

To compare the use of the randomly stopped extreme models with the use of the generalized extreme-value model, we have simulated a sample for $m = 150$ years assuming that N_i is Logarithmic$(p = .95)$ and X_i is Exponential$(\lambda = 0.01)$. We have fitted the true two-parameter $\mathcal{Y}_{Lg-Exp}$ model and the three parameter generalized extreme-value model,

$$\mathcal{Y}_{GEV} = \{Y_{\eta,\theta,\kappa} : F_{Y_{\eta,\theta,\kappa}} = e^{-(1-\kappa(x-\eta)/\theta)^{1/\kappa}}, \eta \in (-\infty,\infty), \theta \in (0,\infty), \kappa \in (-\infty,\infty)\},$$

on this data set by maximum likelihood. We have also fitted the $\mathcal{Y}_{TB2-Exp}$ and $\mathcal{Y}_{ETNB-Exp}$ models, which are the randomly stopped maxima of an exponential sample when the stopping model is the truncated binomial$(2, p)$ and the extended truncated negative binomial (ETNB) with pgf $h_N = \frac{\log(1-pt)^{-r}-1}{\log(1-p)^{-r}-1}$ where r is in $(-1,\infty)$. We also fit $\mathcal{Y}_{PC-LgNor}$, which is the randomly stopped maximum with $\mathcal{N}$ being the potential conjugate model considered in Example 6.1 and $\mathcal{X}$ being the lognormal model.

Table 1 presents the maximum likelihood estimates of the parameters of these five models together with the value of the log-likelihood at its maximum, and their AIC and BIC. Note that the $\mathcal{Y}_{ETNB-Exp}$ model fits the data slightly better than the actual $\mathcal{Y}_{Lg-Exp}$ model, but when $r = 0$ the $\mathcal{Y}_{ETNB-Exp}$ becomes the $\mathcal{Y}_{Lg-Exp}$ model and the likelihood ratio test between these two nested models does not reject the simpler actual model with a $p - val$ of 0.758.

Table 1. *Maximum likelihood parameter estimates, logarithm of the likelihood at the mle, and AIC and BIC for the five models considered for the data on the largest annual rainfall event.*

Model	N.par	MLE			loglikel	AIC	BIC
$\mathcal{Y}_{Lg-Exp}$	2	$\hat{p} = .9574$		$\hat{\lambda} = .0098$	-942.326	1888.65	1894.67
$\mathcal{Y}_{ETNB-Exp}$	3	$\hat{p} = .9844$	$\hat{r} = -.1177$	$\hat{\lambda} = .0108$	-942.172	1890.34	1899.38
$\mathcal{Y}_{TB2-Exp}$	2	$\hat{p} = .3949$		$\hat{\lambda} = .0063$	-945.196	1894.39	1900.41
$\mathcal{Y}_{PC-LgNor}$	3	$\hat{p} = .9352$	$\hat{\mu} = 4.9109$	$\hat{\sigma} = 1.1475$	-952.578	1911.15	1920.19
$\mathcal{Y}_{GEV}$	3	$\hat{\eta} = 129.01$	$\hat{\theta} = 111.25$	$\hat{\kappa} = -.1207$	-954.256	1914.51	1923.54

Even though the $\mathcal{Y}_{GEV}$ model has one more parameter than the actual $\mathcal{Y}_{Lg-Exp}$ model, it fits the simulated data significantly worse than this model, and worse than the other three stopped extreme models tried, even though two of these models assume a wrong stopping model and one of them assumes a wrong stopping and a wrong stopped model. Of course that will not always be the case, and the $\mathcal{Y}_{GEV}$ model will do better than other stopped extreme models, but when one has a good guess on what the stopping and the stopped models could be, the corresponding randomly stopped extreme model will tend to do better than $\mathcal{Y}_{GEV}$.

Note also that an important advantage of using randomly stopped extreme models is that through them one can interpret the estimated parameter values in terms of the parameters of the model for the stopping variable and the parameters of the model for the stopped variable. That provides useful information about the frequency of rain and about the distribution of the amounts of rain in them, which is lacking when the analysis is based on the GEV model.

Remark: When $\mathcal{Y} = \max_{\mathcal{N}}(\mathcal{X})$ one has that $F_Y = h_N(F_X)$ and the pdf of Y is $f_Y = h_N^{'}(F_X) f_X$, where f_X is the pdf of X. Therefore f_Y is a weighted version of f_X and maximizing the likelihood function using data on Y is not any more complicated than doing it with data on X.

3.2. On the usefulness of randomly stopped extreme precursors

Lets assume here that one has data on the magnitude of the strongest earthquake on a given year for m_1 years, $(y_1,\dots,y_{m_1})$, and data on the number of earthquakes in a year for m_2 years, $(n_1,\dots,n_{m_2})$, where the set of years with available data might not coincide. Let's also assume that one has a good model $\mathcal{Y}$ for Y_i and a good model $\mathcal{N}$ for N_i.

Like in the previous example one can pose $Y_i = \max(X_1,\dots,X_{N_i})$ where the magnitudes of the earthquakes, X_j, are i.i.d. realizations of a random variable, X, and hence one can assume that $\mathcal{Y} = \max_{\mathcal{N}}(\mathcal{X})$. In such a setting one might lack data about the X_j and yet the interest in the analysis might be to learn about the distribution of these X_j, and therefore about their cdf, F_X.

In particular, the stopping model for the number of earthquakes, N_i, could for example again be Logarithmic(p), and a good model for the magnitude of the strongest earthquake, Y_i, could be the GEV(η,θ,κ) model that was discarded in the previous example for the largest rainfall. If that was the case the magnitude of earthquakes, X_j, would be a sample from the random variable X that is the N_p-maxprecursor of the GEV r.v., $Y_{\eta,\theta,\kappa}$, and the cdf of X would be:

$$F_{X_{p,\eta,\theta,\kappa}} = F_{\max^{-1}_{N_p}(Y_{\eta,\theta,\kappa})} = h^{-1}_{N_p}(F_{Y_{\eta,\theta,\kappa}}).$$

Hence, by obtaining maximum likelihood estimates of p and of (η,θ,κ) and estimates of their standard deviations using the data on N_i and the data on Y_i one would obtain estimates and confidence intervals for the cdf of X, $\hat{F}_{X_{p,\eta,\theta,\kappa}} = h^{-1}_{N_{\hat{p}}}(F_{Y_{\hat{\eta},\hat{\theta},\hat{\kappa}}})$.

3.3. On the rationale behind using the extensions in Definition 3

Finally, lets assume that in either the hydrology or the seismology settings considered above one guesses that $\mathcal{N}_0$ is the stopping model for the number of events, N_i, and $\mathcal{X}_0$ is the model for the magnitude of the events X_j, but it turns that the statistical model $\mathcal{Y}_0 = \max_{\mathcal{N}_0}(\mathcal{X}_0)$ for $Y_i = \max(X_1,\dots,X_{N_i})$ fails to fit properly the sample of extreme values available, $(y_1,\dots,y_m)$.

In a case like this, if one is confident that $\mathcal{N}_0$ is the right stopping model one will want to extend $\mathcal{Y}_0$ by extending $\mathcal{X}_0$ while still using $\mathcal{N}_0$ as the stopping model. The first model extension in Definition 3 does that by replacing $\mathcal{Y}_0 = \max_{\mathcal{N}_0}(\mathcal{X}_0)$ by:

$$\mathcal{Y}_1 = \max_{\mathcal{N}_0}(\max^{-1}_{\mathcal{N}_0}(\mathcal{Y}_0)) = \{Y_{\xi,\delta_1,\delta_2} : F_{Y_{\xi,\delta_1,\delta_2}} = h_{N_{\delta_2}} \circ h^{-1}_{N_{\delta_1}}(F_{Y_\xi}),\ \xi \in \Xi, \delta_1,\delta_2 \in \mathcal{D}\},$$

where Ξ is the parameter space of $\mathcal{Y}_0$. In this way, the extended model can be posed as $\mathcal{Y}_1 = \max_{\mathcal{N}_0}(\mathcal{X}_1)$ where $\mathcal{X}_0$ has been replaced by its extension, $\mathcal{X}_1 = \max_{\mathcal{N}_0}^{-1}(\max_{\mathcal{N}_0}(\mathcal{X}_0))$. Note that this extension also applies when $\mathcal{Y}_0$ is chosen without making any $\mathcal{X}_0$ explicit, in which case the extended model is $\mathcal{Y}_1 = \max_{\mathcal{N}_0}(\mathcal{X}_1)$ with $\mathcal{X}_1 = \max_{\mathcal{N}_0}^{-1}(\mathcal{Y}_0)$.

By construction, the dimension of the parameter space of $\mathcal{Y}_1 = \max_{\mathcal{N}_0}(\max_{\mathcal{N}_0}^{-1}(\mathcal{Y}_0))$ is never smaller than the one of $\mathcal{Y}_1' = \max_{\mathcal{N}_0}^{-1}(\mathcal{Y}_0)$, which is never smaller than the one of $\mathcal{Y}_0$. This paper investigates when is the initial model always included in the transformed model, and when does repeated use of these extensions fail to keep extending the model.

4. Statistical stability of statistical model transformations

Transformations of a statistical model, $\mathcal{X}$, into a new model, $\mathcal{Y} = \mathcal{T}(\mathcal{X})$, can be classified depending on how initial and final models relate. Most often neither $\mathcal{X}$ nor $\mathcal{Y}$ are included into each other. The next definition distinguishes three possible relationships when they do.

Definition 4. *Let $\mathcal{T}(\cdot)$ transform a statistical model, $\mathcal{X}$, into $\mathcal{Y} = \mathcal{T}(\mathcal{X})$. Then*

1. *if $\mathcal{X} \subset \mathcal{T}(\mathcal{X})$, one says that $\mathcal{X}$ is extended by $\mathcal{T}(\cdot)$, and that $\mathcal{T}(\cdot)$ extends $\mathcal{X}$,*
2. *if $\mathcal{T}(\mathcal{X}) \subset \mathcal{X}$, one says that $\mathcal{X}$ is contracted by $\mathcal{T}(\cdot)$, and that $\mathcal{T}(\cdot)$ contracts $\mathcal{X}$,*
3. *if $\mathcal{T}(\mathcal{X}) = \mathcal{X}$, one says that $\mathcal{X}$ is invariant under $\mathcal{T}(\cdot)$.*

When $\mathcal{X}$ is extended by $\mathcal{T}(\cdot)$ for all $\mathcal{X}$, one says that $\mathcal{T}(\cdot)$ is a model extension. Most often, using a model extension repeatedly will keep extending the model, but some model extensions do not further extend models beyond their first use. These special model extensions are examples of the statistically stable transformations defined next.

Definition 5. *A statistical model transformation, $\mathcal{T}(\cdot)$, is said to be statistically stable if for any model $\mathcal{X}$ one has that $\mathcal{T}(\mathcal{X})$ is invariant under $\mathcal{T}(\cdot)$, and so if $\mathcal{T}(\mathcal{T}(\mathcal{X})) = \mathcal{T}(\mathcal{X})$ for any $\mathcal{X}$.*

When a model transformation is statistically stable, using that transformation twice in a row on any statistical model, $\mathcal{X}$, has the same effect as using it just once.

Definition 5 generalizes to any statistical model transformation the concept of geometric-extreme stability proposed in Marshall and Olkin (1997) in the special case of geometric stopped extreme transformations. Note that the statistical notion of stability presented here is different from probabilistic notions of stability, like the ones used in Rachev and Resnick (1991) or in Fama and Roll (1968), which apply to individual random variables and not to families of them.

The main purpose of the paper is to investigate the properties of the model transformations in Definitions 2 and 3, and to determine when do they work as model extensions, and when are these model extensions statistically stable in the sense of Definition 5. This depends only on the characteristics of the stopping model, and in particular on whether they are extreme auto-reversible and/or closed under pgf composition, the way defined in the next two sections.

5. Stopping models that are extreme reversible or auto-reversible

5.1. Properties of h_N, $\overline{h}_N$, h_N^{-1}, and $\overline{h}_N^{-1}$ for positive count variables

A function, h_N, is the probability generating function of a positive integer-valued random variable N, if and only if it is real valued and such that $h_N(0) = 0$, that $h_N(1) = 1$, and that it is analytic at least on $[0,1)$, with all derivatives in that set being non-negative.

As a consequence, $\overline{h}_N(t) = 1 - h_N(1-t)$ is always such that $\overline{h}_N(0) = 0$, $\overline{h}_N(1) = 1$, and that it is analytic at least on $(0,1]$, with all of its odd derivatives in that set being non-negative, and all of its even derivatives non-positive. If all the moments of N are finite, analyticity and the declared signs of the derivatives of h_N and $\overline{h}_N$ hold at least on $[0,1]$.

From the characterization of h_N it also follows that h_N^{-1} and $\overline{h}_N^{-1}$ are always such that $h_N^{-1}(0) = \overline{h}_N^{-1}(0) = 0$ and $h_N^{-1}(1) = \overline{h}_N^{-1}(1) = 1$, and they are analytic at least on $(0,1)$ with a first derivative that is non-negative in that set. The second derivative of h_N^{-1} is non-positive, while the second derivative of $\overline{h}_N^{-1}$ is non-negative.

In particular, h_N, $\overline{h}_N$, h_N^{-1} and $\overline{h}_N^{-1}$ are always continuous and increasing on $[0,1]$, with h_N and $\overline{h}_N^{-1}$ being convex, and $\overline{h}_N$ and h_N^{-1} being concave.

For the limiting stopping random variable N_I with $\Pr(N_I = 1) = 1$, these four functions coincide, $h_{N_I}(t) = t = \overline{h}_{N_I}(t) = h_{N_I}^{-1}(t) = \overline{h}_{N_I}^{-1}(t)$. The next result will be used later on.

Proposition 1. *If N, N_1 and N_2 are positive integer valued random variables with pgfs h_N, h_{N_1} and h_{N_2}, then 1) $\overline{\overline{h}}_N = h_N$, 2) $\overline{h_N^{-1}} = \overline{h}_N^{-1}$, and 3) $\overline{h}_{N_1} \circ \overline{h}_{N_2} = \overline{(h_{N_1} \circ h_{N_2})}$.*

5.2. Extreme-reversible stopping models

As a consequence of the properties listed above, $\overline{h}_N$ and h_N^{-1} can only be the pgf of a positive integer valued random variable if $N = N_I$.

On the other hand, $\overline{h}_N^{-1}$ sometimes is the pgf of a non-degenerate positive integered random variable, N^*. That leads to the following definition.

Definition 6. *The pair of positive integer valued random variables, (N, N^*), is said to be extreme reversible if $\overline{h}_N^{-1} = h_{N^*}$, and therefore if $h_N^{-1} = \overline{h}_{N^*}$.*

When (N, N^*) are extreme reversible, their pgf's need to be such that:

$$h_{N^*} \circ \overline{h}_N(t) = \overline{h}_{N^*} \circ h_N(t) = t = \overline{h}_N \circ h_{N^*}(t) = h_N \circ \overline{h}_{N^*}(t), \text{ for } t \in [0,1],$$

and in that case, $\max_N^{-1}(X) = \min_{N^*}(X)$, $\min_N^{-1}(X) = \max_{N^*}(X)$, and therefore

$$X = \max_N(\min_{N^*}(X)) = \min_N(\max_{N^*}(X)) = \max_{N^*}(\min_N(X)) = \min_{N^*}(\max_N(X)).$$

It is important to emphasize that extreme reversibility is a property of (N, N^*), and that when it holds, this property applies for any real valued random variable, X.

Example 5.1: For the "potential conjugate" random variable N_b, with $h_{N_b} = 1-(1-t)^b$ for $b \in (0,1]$, one has that $\overline{h}_{N_b}^{-1} = t^{1/b}$, which is a pgf when $b = 1/m$ and m is a positive integer. Hence, for any positive integer m, the N_m with pgf $h_{N_m} = 1-(1-t)^{1/m}$, and N_m^* with pgf $h_{N_m^*} = t^m$, are extreme reversible.

Example 5.2: If N_α is zero-truncated Poisson(α), with:

$$h_{N_\alpha} = \frac{e^{\alpha t}-1}{e^{\alpha}-1}$$

for a given given $\alpha > 0$, then:

$$\overline{h}_{N_\alpha}^{-1} = -\frac{1}{\alpha}\ln(1-(1-e^{-\alpha})t) = h_{N_\alpha^*},$$

which is the pgf of a r.v. N_α^* with a Logarithmic(α) distribution, most often parametrized through $p = 1-e^{-\alpha}$. This means that each zero-truncated Poisson random variable is extreme reversible with one logarithmic random variable.

If a statistical model, $\mathcal{N}^*$, is the set of all random variables N^* that are extreme reversible with a random variable in $\mathcal{N}$, one says that $\mathcal{N}^*$ and $\mathcal{N}$ are a pair of extreme-reversible models.

Note that when $\mathcal{N}$ and $\mathcal{N}^*$ are extreme reversible one has that $\max_{\mathcal{N}}^{-1}(\cdot) = \min_{\mathcal{N}^*}(\cdot)$, and that $\min_{\mathcal{N}}^{-1}(\cdot) = \max_{\mathcal{N}^*}(\cdot)$, and one also has that:

$$\max_{\mathcal{N}}(\max_{\mathcal{N}}^{-1}(\cdot)) = \min_{\mathcal{N}^*}^{-1}(\min_{\mathcal{N}^*}(\cdot)),$$

$$\max_{\mathcal{N}}^{-1}(\max_{\mathcal{N}}(\cdot)) = \min_{\mathcal{N}^*}(\min_{\mathcal{N}^*}^{-1}(\cdot)),$$

and viceversa. As an immediate consequence, when $\mathcal{N}$ and $\mathcal{N}^*$ are extreme-reversible models the set of transformations in Definitions 2 and 3 obtained with $\mathcal{N}$ and the set of transformations in these definitions obtained with $\mathcal{N}^*$ coincide.

5.3. Extreme auto-reversible stopping models

There are instances when N and N^* are the same, hence the next definition.

Definition 7. *The positive integer random variable N is extreme auto-reversible if $\overline{h}_N^{-1} = h_N$, and therefore if $h_N^{-1} = \overline{h}_N$.*

When N is extreme auto-reversible,

$$h_N \circ \overline{h}_N(t) = \overline{h}_N \circ h_N(t) = t, \text{ for } t \in [0,1],$$

which is a condition used in stochastic comparison theorems of Shaked (1975) and Shaked and Wong (1997). When it holds, $\max_N^{-1}(X) = \min_N(X)$, $\min_N^{-1}(X) = \max_N(X)$, and:

$$X = \max_N(\min_N(X)) = \min_N(\max_N(X)).$$

A necessary condition for a r.v. N to be auto-reversible is that $\Pr(N=1) = 1/E[N]$. The next result, providing a way to generate two auto-reversible random variables starting from any pair of reversible ones, will be used to find examples of auto-reversible variables.

Proposition 2. *If (N,N^*) are a pair of extreme-reversible random variables, with pgf's h_N and $h_{N^*} = \overline{h}_N^{-1}$, then the random variables N_1 and N_2, with pgfs $h_{N_1} = h_N \circ h_{N^*}$ and $h_{N_2} = h_{N^*} \circ h_N$, are both extreme auto-reversible.*

Proof: Given that $\overline{h}_N(t) = 1 - h_N(1-t)$, one has that:

$$h_{N_1} \circ \overline{h}_{N_1}(t) = h_N \circ h_{N^*} \circ \overline{h}_N \circ \overline{h}_{N^*}(t) = h_N \circ \overline{h}_{N^*}(t) = t,$$

where the last two steps use the fact that N and N^* are extreme reversible. ■

Corollary 1. *If N is extreme auto-reversible with pgf h_N, then the random variable N_3 with pgf $h_{N_3} = h_N \circ h_N$ is also extreme auto-reversible.*

Example 5.3: Using Proposition 2 with the random variables of Example 5.1 leads to $h_{N_1} = 1-(1-t^m)^{1/m}$ and to $h_{N_2} = (1-(1-t)^{1/m})^m$, which whenever m is a positive integer are the pgf's of two auto-reversible random variables.

Example 5.4: Using Proposition 2 with the random variables of Example 5,2 yields:

$$h_{N_1} = \frac{pt}{1-(1-p)t},$$

for $0 < p = e^{-\alpha} \leq 1$, which is the pgf of the geometric distribution, and

$$h_{N_2} = 1 - (1/\alpha)\log(1 + e^{\alpha} - e^{\alpha t}),$$

for $\alpha > 0$, where N_1 and N_2 are extreme auto-reversible random variables.

When all random variables N in $\mathcal{N}$ are extreme auto-reversible, one says that the stopping model $\mathcal{N}$ is extreme auto-reversible.

When $\mathcal{N}$ is an extreme auto-reversible model one has that $\max_{\mathcal{N}}^{-1}(\cdot) = \min_{\mathcal{N}}(\cdot)$ and that $\min_{\mathcal{N}}^{-1}(\cdot) = \max_{\mathcal{N}}(\cdot)$, and therefore that:

$$\max_{\mathcal{N}}^{-1}(\max_{\mathcal{N}}(\cdot)) = \min_{\mathcal{N}}(\min_{\mathcal{N}}^{-1}(\cdot)) = \min_{\mathcal{N}}(\max_{\mathcal{N}}(\cdot)),$$

$$\min_{\mathcal{N}}^{-1}(\min_{\mathcal{N}}(\cdot)) = \max_{\mathcal{N}}(\max_{\mathcal{N}}^{-1}(\cdot)) = \max_{\mathcal{N}}(\min_{\mathcal{N}}(\cdot)).$$

Therefore, when $\mathcal{N}$ is an extreme auto-reversible model, the four basic and four combined transformations in Definitions 2 and 3 collapse down into two basic and two combined transformations.

6. Stopping models closed under pgf composition

A necessary condition for the transformations in Definitions 2 and 3 to be statistically stable is that the corresponding stopping model be closed under pgf composition as defined next.

Definition 8. *The stopping model* $\mathcal{N} = \{N_\delta : h_{N_\delta},\ \delta \in \mathcal{D}\}$ *is said to be closed under pgf composition, if having* N_{δ_1} *and* N_{δ_2} *with pgfs* $h_{N_{\delta_1}}$ *and* $h_{N_{\delta_2}}$ *belonging to* $\mathcal{N}$*, implies that* N_{δ_3} *with pgf* $h_{N_{\delta_3}} = h_{N_{\delta_1}} \circ h_{N_{\delta_2}}$ *also belongs to* $\mathcal{N}$*.*

Requiring that $\mathcal{N}$ be closed under pgf composition is equivalent to requiring that if N_{δ_1} and N_{δ_2} belong to $\mathcal{N}$, then the N_{δ_1}-stopped sum of N_{δ_2} also belongs to $\mathcal{N}$, and it is thus equivalent to being closed under model compounding.

6.1. Uniparametric stopping models closed under pgf composition

Here we restrict consideration to stopping models, $\mathcal{N} = \{N_\delta : h_{N_\delta} = \sum_{i=1}^{\infty} p_i(\delta)t^i,\ \delta \in \mathcal{D}\}$, that i) are closed under pgf composition, ii) have a parametrization δ such that the $p_i(\delta) = \Pr(N_\delta = i)$ are continuously differentiable in δ for any i, and iii) have a parameter space, $\mathcal{D}$, that is a connected subset of $\mathbb{R}$ with a non-empty interior. From now on, this class of stopping models is denoted in a shorthanded way just as "*models uniparametric and closed under pgf composition*."

By focusing on stopping models continuously differentiable and with this kind of parameter space, we restrict consideration to the kind of stopping models useful in statistical practice. In particular, we essentially require that the parameter space be a non-empty interval, thus avoiding stopping models closed under pgf composition like $\mathcal{N} = \{N_k : h_{N_k} = t^k,\ k \in \mathbb{N}^+\}$, which lead to trivially stable transformations, and we also avoid parameter spaces with isolated points.

The following result, crucial in all that follows, is proved in Appendix 2 (Supplementary material).

Theorem 1. *If a stopping model,* $\mathcal{N} = \{N_\delta :\ \delta \in \mathcal{D}\}$*, is* "uniparametric and closed under pgf composition*" as defined above, then:*

1. $p_1(\delta) = \Pr(N_\delta = 1) > 0$ *for all* $N_\delta \in \mathcal{N}$*,*
2. $\mathcal{N}$ *can be parametrized in an identifiable way through* $\theta = \Pr(N_\delta = 1)$*, or through* $\eta = -\log \Pr(N_\delta = 1)$*, and*

3. *the parameter space is of the form* $(0,\theta_0]$ *for a given* $\theta_0 \leq 1$ *when using* θ*, and it is of the form* $\mathcal{H} = [\eta_0, \infty)$ *for a given* $\eta_0 \geq 0$ *when using* η*.*

From now on, we will always use η as the parametrization for models *uniparametric and closed under pgf composition*. Note that N_I, with $h_{N_I}(t) = t$, belongs to one of these models if, and only if, the lower limit of the parameter space, η_0, is equal to 0.

Next consequence of Theorem 1 relates to repeated use of the transformations in Definition 2.

Theorem 2. *If the stopping model,* $\mathcal{N} = \{N_\eta : h_{N_\eta},\ \eta \in [\eta_0, \infty)\}$*, is "*uniparametric and closed under pgf composition*" as defined above, then:*

1. $h_{N_{\eta_1}} \circ h_{N_{\eta_2}} = h_{N_{\eta_2}} \circ h_{N_{\eta_1}} = h_{N_{\eta_1+\eta_2}}$,
2. $\overline{h}_{N_{\eta_1}} \circ \overline{h}_{N_{\eta_2}} = \overline{h}_{N_{\eta_2}} \circ \overline{h}_{N_{\eta_1}} = \overline{h}_{N_{\eta_1+\eta_2}}$,
3. $h^{-1}_{N_{\eta_1}} \circ h^{-1}_{N_{\eta_2}} = h^{-1}_{N_{\eta_2}} \circ h^{-1}_{N_{\eta_1}} = h^{-1}_{N_{\eta_1+\eta_2}}$,
4. $\overline{h^{-1}}_{N_{\eta_1}} \circ \overline{h^{-1}}_{N_{\eta_2}} = \overline{h^{-1}}_{N_{\eta_2}} \circ \overline{h^{-1}}_{N_{\eta_1}} = \overline{h^{-1}}_{N_{\eta_1+\eta_2}}$,

for all $\eta_1, \eta_2 \in [\eta_0, \infty)$.

Proof: Given that $\mathcal{N}$ is closed under pgf composition, $h_{N_{\eta_1}} \circ h_{N_{\eta_2}} = h_{N_\eta}$, with:

$$\eta = -\log((h_{N_{\eta_1}}(h_{N_{\eta_2}}(t)))'_{|t=0}) = -\log((h'_{N_{\eta_1}}(h_{N_{\eta_2}}(t))h'_{N_{\eta_2}}(t))_{|t=0}) = \eta_1 + \eta_2,$$

and commutativity follows from the commutativity of addition. The other three assertions follow from the fact that, because of Proposition 1,

$$\overline{h}_{N_{\eta_1}} \circ \overline{h}_{N_{\eta_2}} = \overline{(h_{N_{\eta_1}} \circ h_{N_{\eta_2}})} = \overline{h}_{N_{\eta_1+\eta_2}},$$

$$h^{-1}_{N_{\eta_1}} \circ h^{-1}_{N_{\eta_2}} = (h_{N_{\eta_2}} \circ h_{N_{\eta_1}})^{-1} = h^{-1}_{N_{\eta_1+\eta_2}},$$

and

$$\overline{h^{-1}}_{N_{\eta_1}} \circ \overline{h^{-1}}_{N_{\eta_2}} = \overline{(h_{N_{\eta_2}} \circ h_{N_{\eta_1}})^{-1}} = \overline{h^{-1}}_{N_{\eta_1+\eta_2}}.$$

■

The second result that follows from Theorem 1 will imply that under stopping models closed under pgf composition, Definition 3 yields only two distinct extensions, and that the basic transformations in Definition 2 are restricted versions of them.

Theorem 3. *If the stopping model,* $\mathcal{N} = \{N_\eta : h_{N_\eta},\ \eta \in [\eta_0, \infty)\}$*, is "*uniparametric and closed under pgf composition*" as defined above, then:*

$$h^{-1}_{N_{\eta_2}} \circ h_{N_{\eta_1}} = h_{N_{\eta_1}} \circ h^{-1}_{N_{\eta_2}}$$

for all $\eta_1, \eta_2 \in [\eta_0, \infty)$. Furthermore, $H_{\mathcal{N}}(t;\eta_1,\eta_2) = h_{N_{\eta_1}} \circ h^{-1}_{N_{\eta_2}}(t)$ can be parametrized in an identifiable way through $\eta = -\log(H'_{\mathcal{N}}(0;\eta_1,\eta_2)) = \eta_1 - \eta_2$, and if one denotes $H_{\mathcal{N},\eta}(t) = H_{\mathcal{N}}(t;\eta_1,\eta_2)$ with $\eta \in \mathbb{R}$, then

1. *$H_{\mathcal{N},\eta} \circ H_{\mathcal{N},\eta'} = H_{\mathcal{N},\eta+\eta'}$ for all $\eta, \eta' \in \mathbb{R}$,*
2. *when $\eta \geq \eta_0$, then $H_{\mathcal{N},\eta} = h_{N_\eta}$,*
3. *when $\eta \geq \eta_0$, then $H_{\mathcal{N},-\eta} = h^{-1}_{N_\eta}$,*
4. *$H_{\mathcal{N},\eta=0}(t) = t$.*

Likewise, $\overline{h}_{N_{\eta_1}} \circ \overline{h}^{-1}_{N_{\eta_2}} = \overline{h}^{-1}_{N_{\eta_2}} \circ \overline{h}_{N_{\eta_1}}$, and the properties listed above also apply for $\overline{H}_{\mathcal{N},\eta}(t) = \overline{H}_{\mathcal{N}}(t;\eta_1,\eta_2) = \overline{h}_{N_{\eta_1}} \circ \overline{h}^{-1}_{N_{\eta_2}}(t) = 1 - H_{\mathcal{N},\eta}(1-t)$.

Proof: The commutativity for $\eta_1, \eta_2 \in [\eta_0, \infty)$ follows from:

$$h^{-1}_{N_{\eta_2}} \circ h_{N_{\eta_1}} = h^{-1}_{N_{\eta_2}} \circ h_{N_{\eta_1}} \circ h_{N_{\eta_2}} \circ h^{-1}_{N_{\eta_2}} = h^{-1}_{N_{\eta_2}} \circ h_{N_{\eta_2}} \circ h_{N_{\eta_1}} \circ h^{-1}_{N_{\eta_2}} = h_{N_{\eta_1}} \circ h^{-1}_{N_{\eta_2}}.$$

$H_{\mathcal{N}}(t;\eta_1,\eta_2)$ can be parametrized through $\eta = -\log(H'_{\mathcal{N}}(0;\eta_1,\eta_2)) = \eta_1 - \eta_2$ because

$$H'_{\mathcal{N}}(0;\eta_1,\eta_2) = \left(h^{-1}_{N_{\eta_2}}\right)'(0) \cdot h'_{N_{\eta_1}}(0) = \frac{1}{h'_{N_{\eta_2}}(0)} h'_{N_{\eta_1}}(0) = e^{-(\eta_1-\eta_2)},$$

and if $\eta = \eta_1 - \eta_2 = \eta'_1 - \eta'_2$ with $\eta_1, \eta_2, \eta'_1, \eta'_2 \geq \eta_0$, then:

$$h^{-1}_{N_{\eta_2}} \circ h_{N_{\eta_1}} = h^{-1}_{N_{\eta'_2}} \circ h_{N_{\eta'_2}} \circ h^{-1}_{N_{\eta_2}} \circ h_{N_{\eta_1}} = h^{-1}_{N_{\eta'_2}} \circ h_{N_{\eta'_2}} \circ h_{N_{\eta_1}} \circ h^{-1}_{N_{\eta_2}} = h^{-1}_{N_{\eta'_2}} \circ h_{N_{\eta_1}+N_{\eta'_2}} \circ h^{-1}_{N_{\eta_2}} =$$

$$h^{-1}_{N_{\eta'_2}} \circ h_{N_{\eta'_1}+N_{\eta_2}} \circ h^{-1}_{N_{\eta_2}} = h^{-1}_{N_{\eta'_2}} \circ h_{N_{\eta'_1}} \circ h_{N_{\eta_2}} \circ h^{-1}_{N_{\eta_2}} = h^{-1}_{N_{\eta'_2}} \circ h_{N_{\eta'_1}},$$

and because if $h^{-1}_{N_{\eta_2}} \circ h_{N_{\eta_1}} = h^{-1}_{N_{\eta'_2}} \circ h_{N_{\eta'_1}}$ with $\eta_1, \eta_2, \eta'_1, \eta'_2 \geq \eta_0$, then:

$$\left(h^{-1}_{N_{\eta_2}} \circ h_{N_{\eta_1}}\right)'_{|t=0} = \left(h^{-1}_{N_{\eta'_2}} \circ h_{N_{\eta'_1}}\right)'_{|t=0},$$

and so $e^{-(\eta_1-\eta_2)} = e^{-(\eta'_1-\eta'_2)}$ and $\eta_1 - \eta_2 = \eta'_1 - \eta'_2$. To prove the additivity of $H_{\mathcal{N},\eta} \circ H_{\mathcal{N},\eta'}$, let $\beta \geq \eta_0 + \max(|\eta|, |\eta'|)$ and note that:

$$H_{\mathcal{N},\eta} \circ H_{\mathcal{N},\eta'} = (h^{-1}_{N_\beta} \circ h_{N_{\beta+\eta}}) \circ (h^{-1}_{N_\beta} \circ h_{N_{\beta+\eta'}}) =$$

$$h^{-1}_{N_\beta} \circ h^{-1}_{N_\beta} \circ h_{N_{\beta+\eta}} \circ h_{N_{\beta+\eta'}} = h^{-1}_{N_{2\beta}} \circ h_{N_{2\beta+\eta+\eta'}} = H_{\mathcal{N},\eta+\eta'}.$$

Furthermore, letting $\beta \geq \eta_0$ one has that for any $\eta \geq \eta_0$:

$$H_{\mathcal{N},\eta} = h_{N_\beta}^{-1} \circ h_{N_{\beta+\eta}} = h_{N_\beta}^{-1} \circ h_{N_\beta} \circ h_{N_\eta} = h_{N_\eta},$$

$$H_{\mathcal{N},-\eta} = h_{N_{\beta+\eta}}^{-1} \circ h_{N_\beta} = h_{N_\eta}^{-1} \circ h_{N_\beta}^{-1} \circ h_{N_\beta} = h_{N_\eta}^{-1},$$

and that, $H_{\mathcal{N},\eta=0}(t) = h_{N_\beta}^{-1} \circ h_{N_\beta}(t) = t$. ■

By using $H_{\mathcal{N},\eta}$ or $\overline{H}_{\mathcal{N},\eta}$ with $\eta \in \mathbb{R}$ in a model extension of Definition 3, one extends the parameter space through values of η in the whole real line and not just in $\mathcal{H} = [\eta_0, \infty)$.

Many stopping models satisfy the consequences of Theorem 1 without being closed under pgf composition. Next, an extra necessary condition for being a stopping model closed under pgf composition is obtained by imposing that the t^2 coefficients of the series expansion of $h_{N_{\eta_2}}^{-1} \circ h_{N_{\eta_1}}$ and of $h_{N_{\eta_1}} \circ h_{N_{\eta_2}}^{-1}$ have to be equal for any $\eta_1, \eta_2 \in [\eta_0, \infty)$. Imposing that higher order term coefficients of these expansions are equal leads to other necessary conditions.

Corollary 2. *If a stopping model* $\mathcal{N} = \{N_\eta : h_{N_\eta},\ \eta \in [\eta_0, \infty)\}$ *is closed under pgf composition,*

$$\frac{\Pr(N_\eta = 2)}{\Pr(N_\eta = 1)(1 - \Pr(N_\eta = 1))} = C, \ \textit{for all} \ \ \eta \in [\eta_0, \infty).$$

6.2. Examples of stopping models closed under pgf composition

Example 6.1: The *potential conjugate* model,

$$\mathcal{N} = \{N_p : h_{N_p} = 1 - (1-t)^p, \ \ p \in (0,1]\},$$

is closed under pgf composition with $E[N_p] = \infty$ and $\Pr(N_p = 1) = p$, and therefore with $\eta = -\log p \in [0, \infty)$. It includes N_I but it is not auto-reversible as described in Section 5.3.

Example 6.2: The zero-truncated geometric model,

$$\mathcal{N} = \{N_p : h_{N_p} = \frac{pt}{1-(1-p)t}, \ \ p \in (0,1]\},$$

is closed under pgf composition, with $\Pr(N_p = 1) = p$ and $\eta = -\log p \in [0, \infty)$. It includes N_I and, as indicated in Example 5.4, it is auto-reversible.

The next result provides a way of generating a new model closed under pgf composition, starting from an initial model closed under pgf composition and two random variables that do not belong to the initial model but whose pgf composition does.

Proposition 3. *Let the stopping model $\mathcal{N} = \{N_\eta : h_{N_\eta},\ \eta \in [\eta_0, \infty)\}$ be closed under pgf composition, and let N_1 and N_2 be two random variables that do not belong to $\mathcal{N}$ but such that $h_{N_1} \circ h_{N_2} = h_{N_\alpha}$ with $N_\alpha \in \mathcal{N}$. Then, for any given $\alpha > 0$ the statistical model*

$$\mathcal{N}_\alpha = \{\tilde{N}_\eta : h_{\tilde{N}_\eta} = h_{N_2} \circ h_{N_{\eta-\alpha}} \circ h_{N_1},\ \ \eta \in [\alpha + \eta_0, \infty)\}$$

is also closed under pgf composition.

Proof: If $\tilde{N}_{\eta_1}$ and $\tilde{N}_{\eta_2}$ are random variables that belong to $\mathcal{N}_\alpha$, then

$$h_{\tilde{N}_{\eta_1}} \circ h_{\tilde{N}_{\eta_2}} = h_{N_2} \circ h_{N_{\eta_1-\alpha}} \circ h_{N_1} \circ h_{N_2} \circ h_{N_{\eta_2-\alpha}} \circ h_{N_1} =$$

$$h_{N_2} \circ h_{N_{\eta_1-\alpha}} \circ h_{N_\alpha} \circ h_{N_{\eta_2-\alpha}} \circ h_{N_1} = h_{N_2} \circ h_{N_{\eta_1+\eta_2-\alpha}} \circ h_{N_1},$$

which is the pgf of a random variable $\tilde{N}_{\eta_1+\eta_2}$ that also belongs to $\mathcal{N}_\alpha$. ■

Using Proposition 3 twice in a row does not generate any new family of models.

Next, this result is used to generate three families of stopping models closed under pgf composition starting from the geometric model.

Example 6.3: If N_1 is zero-truncated Poisson(α) and N_2 is Logarithmic(α), as in Example 5.2, then $h_{N_1} \circ h_{N_2}$ is the pgf of a Geometric($p = e^{-\alpha}$) and by Proposition 3 one has that for any given value of $\alpha > 0$ the statistical model:

$$\mathcal{N}_\alpha = \{N_\eta : h_{N_\eta} = \frac{1}{\alpha} \ln\left(1 + \frac{(e^{\alpha t} - 1)(e^\alpha - 1)}{(e^\eta - 1)(e^\alpha - e^{\alpha t}) + e^\alpha - 1}\right),\ \ \eta \in [\alpha, \infty)\},$$

is closed under pgf composition. In the limit, when α tends to 0 this model becomes the geometric model, and when α tends to ∞ it becomes N_I. The model $\mathcal{N}_\alpha$ is extreme auto-reversible for every α, but it only includes N_I in the limiting cases mentioned.

Example 6.4: Let N_1 be zero-truncated Negative-Binomial($\alpha\beta, \beta$), with:

$$h_{N_1} = \frac{(1 - (1 - e^{-\frac{\alpha}{\beta}})t)^{-\beta} - 1}{e^\alpha - 1},$$

and let N_2 be extended truncated Negative-Binomial($\alpha, -1/\beta$) in Engen (1974), with:

$$h_{N_2} = \frac{(1 - (1 - e^{-\alpha})t)^{\frac{1}{\beta}} - 1}{e^{-\frac{\alpha}{\beta}} - 1},$$

where $\alpha \geq 0$ and $\beta \geq 1$. Then $h_{N_1} \circ h_{N_2}$ is the pgf of a Geometric($p = e^{-\alpha}$), and by Proposition 3 one has that given any $\alpha \geq 0$ and $\beta \geq 1$ the statistical model:

$$\mathcal{N}_{\alpha,\beta} = \{N_\eta : h_{N_\eta} = \frac{1 - \left(\frac{(-\mathrm{e}^{\alpha+\eta}+1)(1-t+t\mathrm{e}^{-\frac{\alpha}{\beta}})^\beta + \mathrm{e}^\eta - 1}{(\mathrm{e}^\alpha - \mathrm{e}^{\alpha+\eta})(1-t+t\mathrm{e}^{-\frac{\alpha}{\beta}})^\beta + \mathrm{e}^\eta - \mathrm{e}^\alpha}\right)^{\frac{1}{\beta}}}{1 - \mathrm{e}^{-\frac{\alpha}{\beta}}},\ \ \eta \in [\alpha, \infty)\},$$

is closed under pgf composition. When β tends to ∞ one obtains the models in Example 6.3, and when α tends to 0 or β converges to 1 one obtains the geometric model in Example 6.2. Other than in these limiting cases, $\mathcal{N}_{\alpha,\beta}$ is neither extreme auto-reversible, nor includes N_I.

Example 6.5: Let N_1 be zero-truncated Binomial($n, p = 1-e^{-\alpha/n}$), with:

$$h_{N_1} = \frac{(1+(e^{\frac{\alpha}{n}}-1)t)^n - 1}{e^{\alpha}-1},$$

and let N_2 be zero-truncated Negative-Binomial($\alpha, 1/n$), with:

$$h_{N_2} = \frac{(1-(1-e^{-\alpha})t)^{-\frac{1}{n}} - 1}{e^{\frac{\alpha}{n}}-1},$$

where $\alpha \geq 0$ and $n \in \mathbb{N}^+$. Then, $h_{N_1} \circ h_{N_2}$ is the pgf of a Geometric($p = e^{-\alpha}$), and by Proposition 3 one has that for any given $\alpha \geq 0$ and $n \in \mathbb{N}^+$ the statistical model

$$\mathcal{N}_{\alpha,n} = \{N_\eta : h_{N_\eta} = \frac{\left(\frac{(\mathrm{e}^{\eta}-1)(-t+1+t\mathrm{e}^{\frac{\alpha}{n}})^n - \mathrm{e}^{\alpha+\eta}+1}{(-\mathrm{e}^{\alpha}+\mathrm{e}^{\eta})(-t+1+t\mathrm{e}^{\frac{\alpha}{n}})^n + \mathrm{e}^{\alpha} - \mathrm{e}^{\alpha+\eta}}\right)^{-\frac{1}{n}} - 1}{\mathrm{e}^{\frac{\alpha}{n}}-1},\ \eta \in [\alpha, \infty)\},$$

is closed under pgf composition. In the limit, when n converges to ∞ one obtains the models in Example 6.3, and when α converges to 0, or when n is 1, one obtains the geometric model in Example 6.2. Other than in these limiting cases, $\mathcal{N}_{\alpha,n}$ is neither extreme auto-reversible, nor includes N_I, but it is extreme reversible with the $\mathcal{N}_{\alpha,\beta=n}$ in Example 6.4.

The next result provides a way of generating a family of statistical models closed under pgf composition starting from any model that is like that.

Proposition 4. *If the stopping model* $\mathcal{N} = \{N_\eta : h_{N_\eta}(t),\ \eta \in [\eta_0, \infty)\}$ *is closed under pgf composition then, for every given* $k \in \mathbb{N}^+$, *the statistical model*

$$\mathcal{N}_k = \{\tilde{N}_\eta : h_{\tilde{N}_\eta}(t) = \left(h_{N_\eta}(t^k)\right)^{1/k},\ \eta \in [\eta_0, \infty)\}$$

is also closed under pgf composition.

Using this result with Examples 6.1 and 6.2 one has that for every $k \in \mathbb{N}^+$ the models

$$\mathcal{N}_k = \{N_p : h_{N_p} = \left(1-\left(1-t^k\right)^p\right)^{1/k},\ p \in (0,1]\},$$

and

$$\mathcal{N}_k = \{N_p : h_{N_p} = \left(\frac{pt^k}{1-(1-p)t^k}\right)^{1/k},\ p \in (0,1]\},$$

are closed under pgf composition with $\eta = -(1/k)\log p$ and support $n = 1, k+1, 2k+1, \ldots$.

Finally we present a family of statistical models closed under pgf composition that embed Examples 6.1 and 6.2 as limiting cases and all include N_I.

Example 6.6: Given any value $\alpha \in (0,1)$, the statistical model

$$\mathcal{N}_\alpha = \{N_p : h_{N_p} = 1 - \frac{1-t}{(p+(1-p)(1-t)^\alpha)^{1/\alpha}},\ \ p \in (0,1]\},$$

is closed under pgf composition with $E[N_p] = p^{-1/\alpha}$, with $Var[N_p] = \infty$ and with $\eta = -\log p \in [0,\infty)$, and it contains N_I. In the limit, when α tends to 0 one obtains the model in Example 6.1, and when α tends to 1 one obtains the model in Example 6.2.

7. Randomly stopped extreme-based model transformations

7.1. Model transformations in Definitions 2 and 3

Given the properties of h_N and of $\overline{h}_N^{-1}$ it follows that $F_{\max_N(X)}(y) \leq F_X(y)$ and $F_{\min_N^{-1}(X)}(y) \leq F_X(y)$ for all y in their domain, and therefore that $\max_N(X)$ and $\min_N^{-1}(X)$ are random variables always larger than X in the usual stochastic order. Furthermore, given the properties of $\overline{h}_N$ and of h_N^{-1} it follows that $F_{\min_N(X)}(y) \geq F_X(y)$ and $F_{\max_N^{-1}(X)}(y) \geq F_X(y)$, and therefore that $\min_N(X)$ and $\max_N^{-1}(X)$ are always smaller than X in that stochastic order.

Hence, two of the basic transformations of Definition 2 transform any model $\mathcal{X} = \{X_\theta,\ \theta \in \Theta\}$ into a model $\mathcal{Y} = \{Y_{\theta,\delta},\ \theta \in \Theta, \delta \in \mathcal{D}\}$ with random variables $Y_{\theta,\delta}$ stochastically larger than X_θ, while the other two transform $\mathcal{X}$ into a model with $Y_{\theta,\delta}$ stochastically smaller than X_θ.

The four combined transformations of Definition 3 transform $\mathcal{X}$ into a model $\mathcal{Y}$ with random variables $Y_{\theta,\delta_1,\delta_2}$ that can be stochastically larger and smaller than X_θ.

By construction, the dimension of the parameter space of models obtained through transformations in Definition 3 is never smaller than the dimension of the parameter space of models obtained through transformations in Definition 2, which in turn is never smaller than the dimension of the parameter space of the initial model. We next investigate when is the initial model always included in the transformed model, and when does repeated use of these extensions leave the extended model unchanged.

7.2. When do transformations work as extensions?

A sufficient condition for basic transformations in Definition 2 to work as extensions for any model, $\mathcal{X}$, is that the identity belongs to the stopping model.

Proposition 5. *If $N_I \in \mathcal{N}$, with $\Pr(N_I = 1) = 1$, then the four basic model transformations in Definition 2 work as a model extension of $\mathcal{X}$, for any $\mathcal{X}$.*

Proof: If N_I, with $h_{N_I}(t) = t$, belongs to $\mathcal{N}$, then $X \in \mathcal{X}$ implies that $X \in \max_{\mathcal{N}}(\mathcal{X})$, and so $\mathcal{X} \subset \max_{\mathcal{N}}(\mathcal{X})$. The same argument applies to the other three basic transformations. ■

If one starts with a single random variable, $\mathcal{X} = \{X\}$, then $N_I \in \mathcal{N}$ is necessary and sufficient for X to be included in $\max_{\mathcal{N}}(X)$ and in $\min_{\mathcal{N}}(X)$. In general though, one can find instances of specific models, $\mathcal{X}$, included in $\max_{\mathcal{N}}(\mathcal{X})$ or in $\min_{\mathcal{N}}(\mathcal{X})$ without N_I belonging to $\mathcal{N}$.

On the other hand, the four combined mechanisms of Definition 3 always work as model extensions, irrespective of whether N_I is in $\mathcal{N}$ or not.

Proposition 6. *The four model transformation in Definition 3 work as a model extension of $\mathcal{X}$, for any $\mathcal{X}$. That is so, even if one of the two new parameters, δ_1 or δ_2, is fixed.*

Proof: $F_{X_\theta} \in \mathcal{X}$ implies that $F_{Y_{\theta,\delta_1,\delta_2}} = h_{N_{\delta_2}} \circ h^{-1}_{N_{\delta_1}}(F_{X_\theta}) \in \max_{\mathcal{N}}(\max^{-1}_{\mathcal{N}}(\mathcal{X}))$ for all $\delta_1, \delta_2 \in \mathcal{D}$, and in particular $F_{X_\theta} = h_{N_{\delta_1}} \circ h^{-1}_{N_{\delta_1}}(F_{X_\theta}) \in \max_{\mathcal{N}}(\max^{-1}_{\mathcal{N}}(\mathcal{X}))$, which means that $\mathcal{X} \subset \max_{\mathcal{N}}(\max^{-1}_{\mathcal{N}}(\mathcal{X}))$. The same argument applies to the other three transformations, and when any of the two new parameters is fixed. ■

Different from the transformations in Definition 3, using $\max_{\mathcal{N}}(\cdot) \cup \min_{\mathcal{N}}(\cdot)$ with a stopping model $\mathcal{N}$ that does not include N_I does not always work as a model extension.

7.3. When are the extensions statistically stable?

Under general uniparametric stopping models, the basic transformations of Definition 2 usually add one dimension to the parameter space, while the combined transformations of Definition 3 usually add two dimensions to it.

Instead, when the stopping model is uniparametric and closed under pgf composition both basic as well as combined transformations add at most a single dimension, and the basic transformations of Definition 2 become restricted versions of the combined transformations of Definition 3 with the extra parameter, η, of the basic transformations taking values on a semi-line and the extra parameter, η, of the combined transformation taking values on the whole real line.

Furthermore, under general stopping models repeated use of these extensions usually keep extending the models. Instead, when the stopping model is closed under pgf composition and the transformation works as an extension, then it is always a statistically stable extension and hence repeated use of that extension leaves the extended model unchanged.

Proposition 7. *If the stopping model $\mathcal{N} = \{N_\eta : h_{N_\eta},\ \eta \in [\eta_0, \infty)\}$ is "*uniparametric and closed under pgf composition*" then the model transformations in Definition 2 are such that:*

1. *if $\eta_0 = 0$, then $\max_{\mathcal{N}}(\cdot)$, $\min_{\mathcal{N}}(\cdot)$, $\max^{-1}_{\mathcal{N}}(\cdot)$, and $\min^{-1}_{\mathcal{N}}(\cdot)$ are statistical model extensions that are statistically stable, and*

2. *if* $\eta_0 > 0$*, then* $\max_{\mathcal{N}}(\mathcal{X})$ *is contracted by* $\max_{\mathcal{N}}(\cdot)$*,* $\min_{\mathcal{N}}(\mathcal{X})$ *is contracted by* $\min_{\mathcal{N}}(\cdot)$*,* $\max^{-1}_{\mathcal{N}}(\mathcal{X})$ *is contracted by* $\max^{-1}_{\mathcal{N}}(\cdot)$*, and* $\min^{-1}_{\mathcal{N}}(\mathcal{X})$ *is contracted by* $\min^{-1}_{\mathcal{N}}(\cdot)$*, for all* $\mathcal{X}$*.*

Proof: By Theorem 2 one has that for any $\mathcal{X}$,

$$\mathcal{Y} = \max_{\mathcal{N}}(\max_{\mathcal{N}}(\mathcal{X})) = \{Y_{\theta,\eta_1,\eta_2} : F_{Y_{\theta,\eta_1,\eta_2}} = h_{N_{\eta_2}} \circ h_{N_{\eta_1}}(F_{X_\theta}),\ \theta \in \Theta, \eta_1, \eta_2 \in [\eta_0, \infty)\} =$$

$$\{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = h_{N_{\eta=\eta_1+\eta_2}}(F_{X_\theta}),\ \theta \in \Theta, \eta \in [2\eta_0, \infty)\} \subset$$

$$\{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = h_{N_\eta}(F_{X_\theta}),\ \theta \in \Theta, \eta \in [\eta_0, \infty)\} = \max_{\mathcal{N}}(\mathcal{X}),$$

and so if $\eta_0 > 0$, then $\max_{\mathcal{N}}(\cdot)$ contracts $\max_{\mathcal{N}}(\mathcal{X})$. When $\eta_0 = 0$,

$$\max_{\mathcal{N}}(\max_{\mathcal{N}}(\mathcal{X})) = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = h_{N_\eta}(F_{X_\theta}),\ \theta \in \Theta, \eta \in [0, \infty)\} = \max_{\mathcal{N}}(\mathcal{X}),$$

which means that $\max_{\mathcal{N}}(\cdot)$ is a statistically stable extension. The same argument applies to the other three transformations in Definition 2. ■

The next result establishes that under uniparametric stopping models closed under pgf composition, there are only two distinct combined extensions and they are statistically stable.

Proposition 8. *If the stopping model* $\mathcal{N}$ *is "*uniparametric and closed under pgf composition*", then Definition 3 yields only two distinct model extensions which are:*

1. $\mathcal{Y}_1 = \max^{-1}_{\mathcal{N}}(\max_{\mathcal{N}}(\cdot)) = \max_{\mathcal{N}}(\max^{-1}_{\mathcal{N}}(\cdot))$*, and*

2. $\mathcal{Y}_2 = \min^{-1}_{\mathcal{N}}(\min_{\mathcal{N}}(\cdot)) = \min_{\mathcal{N}}(\min^{-1}_{\mathcal{N}}(\cdot))$,

and these two extensions are both statistically stable. Furthermore, in that case it holds that:

1. *the model* $\mathcal{Y}_1 = \max^{-1}_{\mathcal{N}}(\max_{\mathcal{N}}(\mathcal{X}))$ *is invariant under* $\max_{\mathcal{N}}(\cdot)$ *and* $\max^{-1}_{\mathcal{N}}(\cdot)$,

2. *the model* $\mathcal{Y}_2 = \min^{-1}_{\mathcal{N}}(\min_{\mathcal{N}}(\mathcal{X}))$ *is invariant under* $\min_{\mathcal{N}}(\cdot)$ *and* $\min^{-1}_{\mathcal{N}}(\cdot)$*, and*

3. *the transformations in Definition 2 are restricted versions of one of these two extensions.*

Proof: By Theorem 3, one has that for any $\mathcal{X}$,

$$\mathcal{Y}_1 = \max^{-1}_{\mathcal{N}}(\max_{\mathcal{N}}(\mathcal{X})) = \max_{\mathcal{N}}(\max^{-1}_{\mathcal{N}}(\mathcal{X})) = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = H_{\mathcal{N},\eta}(F_{X_\theta}),\ \theta \in \Theta, \eta \in \mathcal{R}\},$$

and the stability follows from that same theorem, because:

$$\max_{\mathcal{N}}(\mathcal{Y}_1) = \{Y_{\theta,\eta+\eta'} : F_{Y_{\theta,\eta+\eta'}} = H_{\mathcal{N},\eta+\eta'}(F_{X_\theta}),\ \theta \in \Theta, \eta + \eta' \in \mathcal{R}\} = \mathcal{Y}_1,$$

$$\max_{\mathcal{N}}^{-1}(\mathcal{Y}_1) = \{Y_{\theta,\eta-\eta'} : F_{Y_{\theta,\eta-\eta'}} = H_{\mathcal{N},\eta-\eta'}(F_{X_\theta}),\ \theta \in \Theta, \eta - \eta' \in \mathcal{R}\} = \mathcal{Y}_1.$$

By Theorem 3 one also has that:

$$\mathcal{Y}_2 = \min_{\mathcal{N}}^{-1}(\min_{\mathcal{N}}(\mathcal{X})) = \min_{\mathcal{N}}(\min_{\mathcal{N}}^{-1}(\mathcal{X})) = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = \overline{H}_{\mathcal{N},\eta}(F_{X_\theta}),\ \theta \in \Theta, \eta \in \mathcal{R}\},$$

where $\overline{H}_{\mathcal{N},\eta}(t) = 1 - H_{\mathcal{N},\eta}(1-t)$, and stability follows likewise. ∎

Corollary 3. *If* $\mathcal{N}$ *is* "uniparametric and closed under pgf composition"*, then:*

1. $\max_{\mathcal{N}}(\mathcal{X}) \cup \max_{\mathcal{N}}^{-1}(\mathcal{X}) \subset \mathcal{Y}_1 = \max_{\mathcal{N}}^{-1}(\max_{\mathcal{N}}(\mathcal{X})) = \max_{\mathcal{N}}(\max_{\mathcal{N}}^{-1}(\mathcal{X}))$,
2. $\min_{\mathcal{N}}(\mathcal{X}) \cup \min_{\mathcal{N}}^{-1}(\mathcal{X}) \subset \mathcal{Y}_2 = \min_{\mathcal{N}}^{-1}(\min_{\mathcal{N}}(\mathcal{X})) = \min_{\mathcal{N}}(\min_{\mathcal{N}}^{-1}(\mathcal{X}))$,

and if $N_I \in \mathcal{N}$*, then the models on the left and the models on the right are equal.*

7.4. What happens with stopping models both closed and extreme reversible?

When two stopping models are closed under pgf composition and extreme reversible, Proposition 8 and the definition of extreme reversibility lead to the next result.

Proposition 9. *If the stopping models* $\mathcal{N}$ *and* $\mathcal{N}^*$ *are uniparametric, closed under pgf composition, and extreme reversible, then the two distinct statistically stable model extensions in Definition 3 obtained with* $\mathcal{N}$ *and the ones obtained with* $\mathcal{N}^*$ *are the same extensions.*

According to Proposition 8, when a stopping model is closed under pgf composition the four extensions in Definition 3 collapse down into two distinct ones. The next result, stating that when a stopping model is both closed and extreme auto-reversible then these two extensions become a single one, is a straight consequence of the definition of extreme-auto-reversibility.

Proposition 10. *If the stopping model* $\mathcal{N}$ *is uniparametric, closed under pgf composition, and extreme auto-reversible, then the four statistically stable model extensions in Definition 3 coincide,*

$$\max_{\mathcal{N}}^{-1}(\max_{\mathcal{N}}(\cdot)) = \max_{\mathcal{N}}(\max_{\mathcal{N}}^{-1}(\cdot)) = \min_{\mathcal{N}}^{-1}(\min_{\mathcal{N}}(\cdot)) = \min_{\mathcal{N}}(\min_{\mathcal{N}}^{-1}(\cdot)),$$

and they coincide with $\min_{\mathcal{N}}(\max_{\mathcal{N}}(\cdot))$ *and with* $\max_{\mathcal{N}}(\min_{\mathcal{N}}(\cdot))$*. If on top of that,* $N_I \in \mathcal{N}$ *(i.e.* $\eta_0 = 0$*), this statistically stable model extension also coincides with* $\max_{\mathcal{N}}(\cdot) \cup \min_{\mathcal{N}}(\cdot)$.

The geometric stopping model satisfies all the conditions of Proposition 10. As a consequence, the Marshall-Olkin extension of $\mathcal{X}$, originally defined to be $\max_{\mathcal{N}}(\mathcal{X}) \cup \min_{\mathcal{N}}(\mathcal{X})$ when $\mathcal{N}$ is geometric, coincides with the extension of $\mathcal{X}$ obtained through Definition 3 with geometric stopping.

As a consequence, we consider the model extensions in Definition 3 to be the natural way to generalize the Marshall-Olkin extension with stopping models other than geometric. Different from what happens if one generalized Marshall-Olkin through $\max_{\mathcal{N}}(\cdot) \cup \min_{\mathcal{N}}(\cdot)$, by generalizing them through the transformations in Definition 3 one guarantees that these transformations will work as model extensions under any stopping model, $\mathcal{N}$.

8. Examples of statistically stable extensions

When one uses the model extensions of Definition 3 with stopping models that are neither closed under pgf composition nor extreme auto-reversible, one obtains four different extensions that are not statistically stable, and the four basic transformations of Definition 2 are not restricted versions of them. As an example, Appendix 1 presents the four basic and the four combined extensions obtained when $\mathcal{N}$ are the zero-truncated Poisson or the logarithmic models.

Here we present the model extensions in Definition 3 obtained when the stopping models are the ones presented in Section 6.2. Given that these stopping models are all closed under pgf composition, all the extensions obtained here are statistically stable in the sense that applying them twice on any given model leads to the same model as applying them once.

Furthermore, because of Proposition 8 another consequence of all these stopping models being closed under pgf composition is that for them Definition 3 yields at most two distinct extensions, and that the transformations in Definition 2 are restrictions of these two extensions and do not need to be considered apart.

In three of the examples, the stopping models are not auto-reversible, and for them the model extension in Definition 3 based on maxima extends $\mathcal{X} = \{X_\theta : F_{X_\theta},\ \theta \in \Theta\}$ into:

$$\mathcal{Y}_1 = \max_{\mathcal{N}}(\max_{\mathcal{N}}^{-1}(\mathcal{X})) = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = H_{\mathcal{N},\eta}(F_{X_\theta}),\ \ \theta \in \Theta, \eta \in (-\infty,\infty)\},$$

and the extension in Definition 3 based on minima extends $\mathcal{X}$ into:

$$\mathcal{Y}_2 = \min_{\mathcal{N}}(\min_{\mathcal{N}}^{-1}(\mathcal{X})) = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = \overline{H}_{\mathcal{N},\eta}(F_{X_\theta}),\ \ \theta \in \Theta, \eta \in (-\infty,\infty)\},$$

with $H_{\mathcal{N},\eta}(\cdot)$ and $\overline{H}_{\mathcal{N},\eta}(\cdot)$ as in Theorem 3.

In the second and third examples the stopping models are auto-reversible, and hence for them these two extension mechanisms, $\mathcal{Y}_1$ and $\mathcal{Y}_2$, coincide because of Proposition 10. The fourth and fifth families of stopping models considered can be reversible, and when they are reversible they lead to the same pair of model extensions because of Proposition 9.

Example 8.1: Let the stopping model be the one in Example 6.1,

$$\mathcal{N} = \{N_\eta : h_{N_\eta} = 1-(1-t)^{e^{-\eta}},\ \eta \in [0,\infty)\},$$

which is not extreme auto-reversible but it includes N_I.

The extension of $\mathcal{X}$ obtained through maxima and precursors of maxima is:

$$\mathcal{Y}_1 = \max_{\mathcal{N}}(\max_{\mathcal{N}}^{-1}(\mathcal{X})) = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = 1-(1-F_{X_\theta})^{e^{-\eta}},\ \theta \in \Theta, \eta \in (-\infty,\infty)\},$$

which is a special case of the extension in Cordeiro and Castro (2009). When one restricts $\eta \geq 0$, here one obtains $\mathcal{Y}_1' = \max_{\mathcal{N}}(\mathcal{X})$, and when one restricts $\eta \leq 0$ one obtains $\mathcal{Y}_1'' = \max_{\mathcal{N}}^{-1}(\mathcal{X})$, and therefore in this case $\mathcal{Y}_1 = \max_{\mathcal{N}}(\mathcal{X}) \cup \max_{\mathcal{N}}^{-1}(\mathcal{X})$.

When $\mathcal{X}$ is for example an exponential random variable, $\mathcal{Y}_1$ becomes the exponential model. Because of the stability of this extension, using it again, now on the exponential model, will leave that model unchanged which means that the exponential model is invariant under this extension. On the other hand, if $\mathcal{X}$ is the logistic model, then $\mathcal{Y}_1$ is the type II generalized logistic model, which will also be invariant under this extension.

In general, when a statistical model is invariant under an extension that is stable, it is because that model can be obtained as the extension of a submodel of it.

The extension of $\mathcal{X}$ obtained through minima and precursors of minima is:

$$\mathcal{Y}_2 = \min_{\mathcal{N}}(\min_{\mathcal{N}}^{-1}(\mathcal{X})) = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = (F_{X_\theta})^{e^{-\eta}},\ \theta \in \Theta, \eta \in (-\infty,\infty)\},$$

which is a special case of the extension in Cordeiro, Ortega and Cunha (2013).

When one restricts $\eta \geq 0$, one obtains $\mathcal{Y}_2' = \min_{\mathcal{N}}(\mathcal{X})$, and when one restricts $\eta \leq 0$ one obtains $\mathcal{Y}_2'' = \min_{\mathcal{N}}^{-1}(\mathcal{X})$, and therefore here $\mathcal{Y}_2 = \min_{\mathcal{N}}(\mathcal{X}) \cup \min_{\mathcal{N}}^{-1}(\mathcal{X})$.

In this case, if $\mathcal{X}$ is for example the Gumbel model with the location parameter fixed, then $\mathcal{Y}_2$ is the two parameter Gumbel model, and because of the stability of this extension the two parameter Gumbel model model will be invariant under this extension. On the other hand, when $\mathcal{X}$ is the logistic model then $\mathcal{Y}_2$ becomes the type I generalized logistic model which by stability will also be invariant under this extension.

Example 8.2: Let the stopping model be the zero-truncated geometric in Example 6.2,

$$\mathcal{N} = \{N_\eta : h_{N_\eta} = \frac{t}{(1-t)e^\eta + t},\ \eta \in [0,\infty)\},$$

which is extreme auto-reversible and includes N_I.

As a consequence of this auto-reversibility the extension of $\mathcal{X}$ obtained through maxima and their precursors or through minima and their precursors here coincide, and it is:

$$\mathcal{Y} = \mathcal{Y}_1 = \mathcal{Y}_2 = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = \frac{F_{X_\theta}}{(1-F_{X_\theta})e^\eta + F_{X_\theta}},\ \theta \in \Theta, \eta \in (-\infty,\infty)\},$$

which is the Marshall-Olkin extension of $\mathcal{X}$. There is a huge literature using this model extension. Here, when one restricts $\eta \geq 0$ one obtains $\mathcal{Y}' = \max_{\mathcal{N}}(\mathcal{X}) = \min_{\mathcal{N}}^{-1}(\mathcal{X})$, and when one restricts $\eta \leq 0$ one obtains $\mathcal{Y}'' = \min_{\mathcal{N}}(\mathcal{X}) = \max_{\mathcal{N}}^{-1}(\mathcal{X})$. As a consequence, this is the only example considered here where $\mathcal{Y} = \max_{\mathcal{N}}(\mathcal{X}) \cup \min_{\mathcal{N}}(\mathcal{X})$.

When for example $\mathcal{X}$ is the logistic model with the location parameter fixed the extended model, $\mathcal{Y}$, is the two parameter logistic model. Because of statistical stability of this extension, applying it again, now on the two-parameter logistic model, leaves the model unchanged, which means that this two parameter model is invariant under this extension.

Example 8.3: Let the stopping model be the $\mathcal{N}_\alpha$ in Example 6.3 for a given $\alpha \geq 0$. Like the geometric model, this one is also extreme auto-reversible, but it only includes N_I when $\alpha = 0$, which is when it becomes the geometric model.

As a consequence of this auto-reversibility, the extensions of $\mathcal{X}$ obtained through maxima and their precursors and the ones obtained through minima and their precursors coincide and are:

$$\mathcal{Y}_\alpha = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = \frac{1}{\alpha} \ln\left(1 + \frac{(e^{\alpha F_{X_\theta}} - 1)(e^\alpha - 1)}{(e^\eta - 1)(e^\alpha - e^{\alpha F_{X_\theta}}) + e^\alpha - 1}\right),\ \theta \in \Theta, \eta \in (-\infty, \infty)\},$$

with $\mathcal{Y}'_\alpha = \max_{\mathcal{N}_\alpha}(\mathcal{X}) = \min_{\mathcal{N}_\alpha}^{-1}(\mathcal{X})$ when one restricts $\eta \geq \alpha$, and with $\mathcal{Y}''_\alpha = \min_{\mathcal{N}_\alpha}(\mathcal{X}) = \max_{\mathcal{N}_\alpha}^{-1}(\mathcal{X})$ when one restricts $\eta \leq -\alpha$. When one restricts $\eta \in (-\alpha, \alpha)$ one obtains $\mathcal{Y}'''_\alpha = \max_{\mathcal{N}_\alpha}(\max_{\mathcal{N}_\alpha}^{-1}(\mathcal{X})) = \max_{\mathcal{N}_\alpha}(\min_{\mathcal{N}_\alpha}(\mathcal{X}))$ with η_1, η_2 such that $\eta_2 - \eta_1 \in (-\alpha, \alpha)$, but this restricted transformation does not coincide with any of the transformations in Definition 2.

Different from what happens under the geometric model with $\alpha = 0$, when $\alpha > 0$ neither $\max_{\mathcal{N}_\alpha}(\mathcal{X})$ nor $\min_{\mathcal{N}_\alpha}(\mathcal{X})$ work as a model extension of $\mathcal{X}$, and $\max_{\mathcal{N}_\alpha}(\mathcal{X}) \cup \min_{\mathcal{N}_\alpha}(\mathcal{X}) \subset \mathcal{Y}_\alpha$ with an inclusion often strict. Hence this is an example where $\max_{\mathcal{N}_\alpha}(\mathcal{X}) \cup \min_{\mathcal{N}_\alpha}(\mathcal{X})$ does not work as a model extension of $\mathcal{X}$, but where using the $\mathcal{Y}_\alpha$ from Definition 3 does.

Example 8.4: Let the stopping model be the $\mathcal{N}_{\alpha,\beta}$ in Example 6.4 for a given $\alpha \geq 0$ and $\beta \geq 1$. Here N_I is not in the model, and the model is not auto-reversible and therefore it yields two different model extension mechanisms.

The extension of $\mathcal{X}$ obtained through maxima and their precursors is:

$$\mathcal{Y}_{1_{\alpha,\beta}} = \{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = \frac{1 - \left(\frac{(1-\mathrm{e}^{\alpha+\eta})\left(1-F_{X_\theta}\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}}\right)\right)^\beta + \mathrm{e}^\eta - 1}{(\mathrm{e}^\alpha - \mathrm{e}^{\alpha+\eta})\left(1-F_{X_\theta}\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}}\right)\right)^\beta + \mathrm{e}^\eta - \mathrm{e}^\alpha}\right)^{\frac{1}{\beta}}}{1 - \mathrm{e}^{-\frac{\alpha}{\beta}}},\ \theta \in \Theta, \eta \in (-\infty, \infty)\},$$

and here one obtains $\mathcal{Y}'_{1_{\alpha,\beta}} = \max_{\mathcal{N}_{\alpha,\beta}}(\mathcal{X})$ when $\eta \geq \alpha$, and $\mathcal{Y}''_{1_{\alpha,\beta}} = \max_{\mathcal{N}_{\alpha,\beta}}^{-1}(\mathcal{X})$ when $\eta \leq -\alpha$. When $\eta \in (-\alpha, \alpha)$ one has that $\mathcal{Y}'''_{1_{\alpha,\beta}} = \max_{\mathcal{N}_{\alpha,\beta}}(\max_{\mathcal{N}_{\alpha,\beta}}^{-1}(\mathcal{X}))$ with η_1, η_2 such that $\eta_2 - \eta_1 \in (-\alpha, \alpha)$, which can neither be obtained through $\max_{\mathcal{N}_{\alpha,\beta}}(\cdot)$ nor through $\max^{-1}_{\mathcal{N}_{\alpha,\beta}}(\cdot)$.

The model extension of $\mathcal{X}$ obtained through minima and their precursors is:

$$\mathcal{Y}_{2_{\alpha,\beta}} = \left\{Y_{\theta,\eta} : F_{Y_{\theta,\eta}} = \frac{1 - \left(\frac{(\mathrm{e}^{\eta}-\mathrm{e}^{\alpha})\left(1+F_{X_\theta}\left(\mathrm{e}^{\frac{\alpha}{\beta}}-1\right)\right)^{\beta}+\mathrm{e}^{\alpha}-\mathrm{e}^{\alpha+\eta}}{(\mathrm{e}^{\eta}-1)\left(1+F_{X_\theta}\left(\mathrm{e}^{\frac{\alpha}{\beta}}-1\right)\right)^{\beta}-\mathrm{e}^{\eta+\alpha}+1}\right)^{\frac{1}{\beta}}}{1-\mathrm{e}^{\frac{\alpha}{\beta}}},\ \theta \in \Theta, \eta \in (-\infty, \infty)\right\},$$

and one obtains $\mathcal{Y}'_{2_{\alpha,\beta}} = \min_{\mathcal{N}_{\alpha,\beta}}(\mathcal{X})$ when $\eta \geq \alpha$, and $\mathcal{Y}''_{2_{\alpha,\beta}} = \min^{-1}_{\mathcal{N}_{\alpha,\beta}}(\mathcal{X})$ when $\eta \leq -\alpha$. When $\eta \in (-\alpha, \alpha)$ one has that $\mathcal{Y}'''_{2_{\alpha,\beta}} = \min_{\mathcal{N}_{\alpha,\beta}}(\min^{-1}_{\mathcal{N}_{\alpha,\beta}}(\mathcal{X}))$ with η_1, η_2 such that $\eta_2 - \eta_1 \in (-\alpha, \alpha)$, which can neither be obtained through $\min_{\mathcal{N}_{\alpha,\beta}}(\cdot)$ nor through $\min^{-1}_{\mathcal{N}_{\alpha,\beta}}(\cdot)$.

Here $\max_{\mathcal{N}_{\alpha,\beta}}(\mathcal{X}) \cup \max^{-1}_{\mathcal{N}_{\alpha,\beta}}(\mathcal{X}) \subset \mathcal{Y}_{1\alpha,\beta}$, and $\min_{\mathcal{N}_{\alpha,\beta}}(\mathcal{X}) \cup \min^{-1}_{\mathcal{N}_{\alpha,\beta}}(\mathcal{X}) \subset \mathcal{Y}_{2\alpha,\beta}$ with these inclusions being most often strict.

Example 8.5: Let the stopping model be the $\mathcal{N}_{\alpha,n}$ in Example 6.5 for a given $\alpha \geq 0$ and $n \in \mathbb{N}^+$. This model does not include N_I and it is not extreme auto-reversible, but it is reversible with the $\mathcal{N}_{\alpha,\beta=n}$ in Example 6.4. As a consequence, the model extension obtained with $\mathcal{N}_{\alpha,n}$ through maxima and precursors of maxima, coincide with the model extension obtained with the $\mathcal{N}_{\alpha,\beta=n}$ of Example 6.4 through minima and precursors of minima, and viceversa.

9. Final comments

The main contribution of this article is putting together a set of new concepts needed to define and untangle the properties of a large family of statistical model transformation mechanisms that lead to statistical models useful for the analysis of extreme-value data and in reliability. The concepts introduced are:

1. the notion of N-extreme precursors, which can be understood as the inverse of N-stopped maxima and minima, and the model extension mechanisms derived from them (Definitions 1, 2 and 3), which help generalize Marshall-Olkin extensions beyond geometric stopping,

2. the concept of statistical stability of a statistical model extension (Definition 5), which applies to any statistical model extension and not just to the ones considered in this paper,

3. the idea of extreme reversible and auto-reversible stopping models (Definitions 6 and 7), under which the extensions based on randomly stopped maxima and their

inverses coincide with the extensions based on randomly stopped minima and their inverses,

4. and the idea of stopping models closed under pgf composition (Definition 8), which are the ones leading to statistically stable randomly stopped extreme type of extensions.

All these new concepts are needed for the picture to be complete. In particular, if we touch on methods to generate stopping models that are auto-reversible and/or closed under pgf composition other than the geometric model, it is to help understand that the role played by geometric stopping is not as unique as one might think after reading Marshall Olkin (1997).

A second contribution of this article are a set of theoretical results stating that uniparametric stopping models closed under pgf composition can always be parametrized through $\theta = \Pr(N = 1)$ with a parameter space of the form $(0, \theta_0]$ (Theorem 1), and that the pgfs of these models commute under composition among themselves and with their inverses (Theorems 2 and 3). These results are then used in Section 7 to determine conditions leading to statistically stable extensions.

Only two of the families of statistically stable model extensions presented in Section 8 are based on stopping models that are both closed under pgf composition and extreme auto-reversible. And the geometric model is the only stopping model that we know that shares these two features and includes N_I. Nevertheless, note that in order to obtain statistically stable extensions through Definition 3, one only needs that the stopping model be closed under pgf composition.

The only consequence of using stopping models that, unlike the geometric model, are not extreme auto-reversible is that the extension based on maxima and their inverses does not coincide with the extension based on minima and their inverses, and using stopping models that, unlike geometric, do not include N_I does neither affect the statistical stability nor the fact that the transformations presented in Definition 3 always work as an extension.

Finally, note that our definition of statistical stability is extremely basic and fundamental. A statistical model transformation is statistically stable only if using that transformation twice in a row on any statistical model has the same effect as using that transformation just once. The only reason that we can think for not finding the notion of statistical stability anywhere in the statistical literature is that it might be difficult to prove results of that kind outside the specific context of randomly stopped extreme transformations, and the closely related area of randomly stopped sum transformations; It is easy to check that stopping models closed under pgf composition also lead to randomly stopped sum model extensions that are statistically stable.

We consider statistical stability to be a property that should be central in the study of any type of statistical model extension and not just in the study of the specific extensions considered here, and we intend to keep investigating that.

Acknowledgments

We are grateful for the comments made by the reviewers, which helped us improve this manuscript. This work was supported by the grant PID2021-125380OB-I00 funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe.

References

Arnold, B.C., Balakrishnan, N., Nagaraja, H.N. (1992). *A First Course in Order Statistics*. New York, Wiley.

Consul, P.C. (1984). On the distributions of order statistics for a random sample size. *Statistica Neerlandica*, 38, 249-256.

Cordeiro, G.M., Castro, M. (2009). A new family of generalized distributions. *Journal of Statistical Computation & Simulation*, 81, 883-898.

Cordeiro, G.M., Ortega, E.M.M., Cunha, D.C.C. (2013). The exponential generalized class of distributions. *Journal of Data Science*, 11, 1-27.

Engen, (1974). On species frequency models. *Biometrika*, 61, 263-270.

Fama, E.F., Roll, R. (1968). Some properties of symmetric stable distributions. *Journal of the American Statistical Association*, 63, 817-836.

Gupta, D., Gupta, R.C. (1984). On the distributions of order statistics for a random sample size. *Statistica Neerlandica*, 38, 13-19.

Louzada, F., Beret, E.M.P, Franco, M. A. P. (2012). On the distribution of the minimum or maximum of a random number of i.i.d. lifetime random variables. *Applied Mathematics*, 3, 350-353.

Marshall, A.W., Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84, 641-652.

Rachev, S.T., Resnick, S. (1991). Max-geometric infinite divisibility and stability. *Communications in Statistics. Stochastic Models,* 7, 191-218.

Raghunandanan, K. and Patil, S.A. (1972). On order statistics for random sample size. *Statistica Neerlandica*, 26, 121-126.

Rohatgi, V.K. (1987). Distribution of order statistics with random sample size. *Communications in Statistics. Theory and Methods*, 16, 3739-3743.

Shaked, M. (1975). On the distribution of the minimum and of the maximum of a random number of i.i.d. random variables. In *Statistical Distributions in Scientific Work*. Vol. I. ed. G.P. Patil, S. Kotz and J.K. Ord. Reidel, Dordrecht. pp. 363-380.

Shaked, M., Wong, T. (1997). Stochastic comparisons of random minima and maxima. *Journal of Applied Probability*, 34, 420-425

Appendix 1: Model extensions when $\mathcal{N}$ is the zero truncated Poisson or the logarithmic model

The zero-truncated Poisson(α) model is defined through the set of pgfs:

$$\mathcal{N} = \{N_\alpha : h_{N_\alpha} = \frac{e^{\alpha t}-1}{e^\alpha - 1},\ \ \alpha \in [0,\infty)\}.$$

This model includes N_I and therefore both the basic transformations in Definition 2 as well as the combined transformations in Definition 3 are extensions, but this model is neither extreme auto-reversible, because $\Pr[N_\alpha = 1] = 1/E[N_\alpha]$, nor closed under pgf composition, because

$$\Pr(N_\alpha = 1) = \frac{\alpha}{e^\alpha - 1},$$

$$\Pr(N_\alpha = 2) = \frac{1}{2}\frac{\alpha^2}{e^\alpha - 1},$$

and therefore it does not satisfy the necessary condition of Corollary 2 for being closed,

$$\frac{\Pr(N_\alpha = 2)}{\Pr(N_\alpha = 1)(1-\Pr(N_\alpha = 1))} = \frac{\alpha}{2} + \frac{1}{2}\frac{\alpha^2}{e^\alpha - \alpha - 1} = Constant.$$

The four basic extensions of $\mathcal{X} = \{X_\theta : F_{X_\theta},\ \theta \in \Theta\}$ obtained through Definition 2 are,

$$\max_{\mathcal{N}}(\mathcal{X}) = \{Y_{\theta,\alpha} : F_{Y_{\theta,\alpha}} = \frac{e^{\alpha F_{X_\theta}}-1}{e^\alpha - 1},\ \theta \in \Theta, \alpha \in [0,\infty)\},$$

$$\max_{\mathcal{N}}^{-1}(\mathcal{X}) = \{Y_{\theta,\alpha} : F_{Y_{\theta,\alpha}} = \frac{\ln(1+(e^\alpha - 1)F_{X_\theta})}{\alpha},\ \theta \in \Theta, \alpha \in [0,\infty)\},$$

$$\min_{\mathcal{N}}(\mathcal{X}) = \{Y_{\theta,\alpha} : F_{Y_{\theta,\alpha}} = \frac{e^\alpha(1-e^{-\alpha F_{X_\theta}})}{e^\alpha - 1},\ \theta \in \Theta, \alpha \in [0,\infty)\},$$

$$\min_{\mathcal{N}}^{-1}(\mathcal{X}) = \{Y_{\theta,\alpha} : F_{Y_{\theta,\alpha}} = 1 - \frac{\ln(1+(e^\alpha - 1)(1-F_{X_\theta}))}{\alpha},\ \theta \in \Theta, \alpha \in [0,\infty)\},$$

and the four combined extensions of $\mathcal{X}$ obtained through Definition 3 are,

$$\max_{\mathcal{N}}(\max_{\mathcal{N}}^{-1}(\mathcal{X})) =$$

$$\{Y_{\theta,\alpha_1,\alpha_2} : F_{Y_{\theta,\alpha_1,\alpha_2}} = \frac{1}{\mathrm{e}^{\alpha_2}-1}\left((1+(\mathrm{e}^{\alpha_1}-1)\,F_{X_\theta})^{\frac{\alpha_2}{\alpha_1}} - 1\right),\ \theta \in \Theta, \alpha_1, \alpha_2 \in [0,\infty)\},$$

$$\max_{\mathcal{N}}^{-1}(\max_{\mathcal{N}}(\mathcal{X})) =$$

$$\{Y_{\theta,\alpha_1,\alpha_2} : F_{Y_{\theta,\alpha_1,\alpha_2}} = \frac{1}{\alpha_2}\ln\left(1+\frac{(\mathrm{e}^{\alpha_2}-1)\left(\mathrm{e}^{\alpha_1 F_{X_\theta}}-1\right)}{\mathrm{e}^{\alpha_1}-1}\right),\ \theta \in \Theta, \alpha_1, \alpha_2 \in [0,\infty)\},$$

$$\min_{\mathcal{N}}(\min_{\mathcal{N}}^{-1}(\mathcal{X})) =$$

$$\{Y_{\theta,\alpha_1,\alpha_2} : F_{Y_{\theta,\alpha_1,\alpha_2}} = \frac{\mathrm{e}^{\alpha_2}}{\mathrm{e}^{\alpha_2}-1}\left(1-\left(1-\left(1-\mathrm{e}^{-\alpha_1}\right)F_{X_\theta}\right)^{\frac{\alpha_2}{\alpha_1}}\right),\ \theta\in\Theta, \alpha_1,\alpha_2\in[0,\infty)\},$$

$$\min\nolimits_{\mathcal{N}}^{-1}(\min\nolimits_{\mathcal{N}}(\mathcal{X})) =$$

$$\{Y_{\theta,\alpha_1,\alpha_2} : F_{Y_{\theta,\alpha_1,\alpha_2}} = 1-\frac{1}{\alpha_2}\ln\left(\frac{(\mathrm{e}^{\alpha_2}-1)\left(\mathrm{e}^{\alpha_1\left(1-F_{X_\theta}\right)}-1\right)}{\mathrm{e}^{\alpha_1}-1}+1\right),\ \theta\in\Theta, \alpha_1,\alpha_2\in[0,\infty)\}.$$

Furthermore, according to Example 5.2 the Logarithmic(p) model defined through:

$$\mathcal{N} = \{N_p : h_{N_p} = \frac{\log(1-pt)}{\log(1-p)},\ \ p\in[0,1)\},$$

where $p = 1-e^{-\alpha}$, is extreme reversible with the zero-truncated Poisson model. As a consequence of that property, the set of model extensions obtained through Definitions 2 and 3 using the Logarithmic(p) model coincide with the set of extensions obtained using the zero-truncated Poisson model presented in this Appendix.

The specific extensions for the Logarithmic(p) model are the ones listed above for the truncated Poisson model after replacing α by $-\log(1-p)$, and after switching $\max_{\mathcal{N}}$ and $\min_{\mathcal{N}}^{-1}$ and switching $\min_{\mathcal{N}}$ and $\max_{\mathcal{N}}^{-1}$. For example the $\max_{\mathcal{N}}(\cdot)$ transformation when $\mathcal{N}$ is Logaritmic(p) is the $\min_{\mathcal{N}'}^{-1}(\cdot)$ transformation when $\mathcal{N}'$ is truncated Poisson$(\alpha = -\log(1-p))$, the $\min_{\mathcal{N}}(\cdot)$ transformation is the $\max_{\mathcal{N}'}^{-1}(\cdot)$ transformation, and so on.