Supplemental material for "On statistical model extensions based on randomly stopped extremes"

Jordi Valero and Josep Ginebra[1]

January 2025

The material contained herein is supplementary to the article named in the title and published in SORT-Statistics and Operations Research Transactions Volume 49(1).

[1] Address for correspondence: Department of Statistics and O.R., Polytechnic University of Catalonia, Avgda. Diagonal 647, 6ª Planta, 08028 Barcelona, Spain (E-mail: josep.ginebra@upc.es)

Appendix 2: Proof of Theorem 1 in Section 6

Let $\mathcal{N} = \{N_\delta : h_{N_\delta} = \sum_{i=1}^{\infty} p_i(\delta)t^i,\ \delta \in \mathcal{D}\}$, with $p_i(\delta) = \Pr(N_\delta = i)$, be a uniparametric stopping model closed under pgf composition and with a continuously differentiable parametrization and a connected parameter space with a non-empty interior. That is, assume that the stopping model, besides being closed under pgf composition is also such that $p_i(\delta)$ is continuously differentiable in δ for any i, and that $\mathcal{D}$ is a non-empty interval of $\mathbb{R}$, which is what in the paper is denoted in short as a model "uniparametric and closed under pgf composition."

Let $k_1 = \min\{k : \exists\delta \in \mathcal{D}$ such that $p_k(\delta) > 0\}$, and let $\mathcal{D}_k = \{\delta \in \mathcal{D} :\ p_k(\delta) > 0$ and $p_j(\delta) = 0$ for all $j < k\}$ and $\mathcal{N}_k = \{N_\delta \in \mathcal{N} :\ \delta \in \mathcal{D}_k\}$ be partitions of $\mathcal{D}$ and $\mathcal{N}$. Even though it is assumed that $\mathcal{N} = \cup_{k=1}^{\infty}\mathcal{N}_k$ is closed under pgf composition and $\mathcal{D} = \cup_{k=1}^{\infty}\mathcal{D}_k$ is connected, the $\mathcal{N}_k$ do not have to be closed under pgf composition and the $\mathcal{D}_k$ do not have to be connected.

Let $N_{\tilde{\delta}} = N_{\delta_1} \circ N_{\delta_2}$ stand for the random variable in $\mathcal{N}$ with pgf $h_{N_{\tilde{\delta}}} = h_{N_{\delta_1}} \circ h_{N_{\delta_2}}$, and denote $\tilde{\delta} = \delta_1 \circ \delta_2$. Note that if $N_{\delta_1} \in \mathcal{N}_{k_1}$ and $N_{\delta_2} \in \mathcal{N}_{k_2}$, then $N_{\tilde{\delta}} = N_{\delta_1} \circ N_{\delta_2} \in \mathcal{N}_{k_1k_2}$.

Let $\mathcal{N}^{\circ m} = \{N_{\delta_1} \circ N_{\delta_2} \circ \cdots \circ N_{\delta_m} :\ N_{\delta_j} \in \mathcal{N}\}$, where $m \in \mathbb{N}$, and let $N_\delta^{\circ m} = N_{\delta^{\circ m}} = N_\delta \circ N_\delta \circ \cdots \circ N_\delta$. Note that if $\mathcal{N}$ is uniparametric and closed under pgf composition, then $\mathcal{N}^{\circ m} \subset \mathcal{N}$, and $\mathcal{N}^{\circ m}$ is also a uniparametric and closed under pgf composition.

Next we state and prove four lemmas required to prove Theorem 1.

Stating that "$\Pr(N_\delta = 1) > 0$ for all $N_\delta \in \mathcal{N}$", as in the first point of Theorem 1, is equivalent to stating that "$\mathcal{D}_1 = \mathcal{D}$", because if $k_1 > 1$, or if $k_1 = 1$ but there exists a $\delta_0 \in \mathcal{D}$ such that $p_{k_1}(\delta_0) = 0$, both statements are false, and otherwise they are both true.

Theorem 1 will be proven by *reductio ad absurdum*, and therefore one needs to know how would the stopping models be if they were "uniparametric and closed under pgf composition, but $\Pr(N_\delta = 1) = 0$ for some $N_\delta \in \mathcal{N}$", and hence if "$\mathcal{D}_1 \subsetneq \mathcal{D}$". That is the purpose of the first lemma.

Lemma 1. *If $\mathcal{N}$ is "uniparametric and closed under pgf composition, but such that $\Pr(N_\delta = 1) = 0$ for some $N_\delta \in \mathcal{N}$", then at the boundary of any connected component of $\mathcal{D}_{k_1}$, denoted $\mathcal{D}_{k_1}^*$, there is a δ_0^*, from $\mathring{\mathcal{D}}$, with $p_{k_1}(\delta_0^*) = 0$ and such that if a sequence δ_i in $\mathcal{D}_{k_1}^*$ converges to δ_0^*, then $p_{k_1}(\delta_i)$ converges to 0.*

Proof: Under the conditions stated in the lemma, there exists a δ_0 in $\mathcal{D}$ such that $p_1(\delta_0) = 0$, and so such that if $\delta \in \mathcal{D}_{k_1}^*$ then $N_{\tilde{\delta}} = N_{\delta_0} \circ N_\delta$ belongs to $\mathcal{N}_k$ with $k > k_1$. As a consequence, $p_{k_1}(\tilde{\delta}) = 0$ and therefore $\mathcal{D}_{k_1} \subsetneq \mathcal{D}$, or what is equivalent, $\mathcal{D} \setminus \mathcal{D}_{k_1} \neq \emptyset$. Given that for any δ in $\mathcal{D} \setminus \mathcal{D}_{k_1}$ one has that $p_{k_1}(\delta) = 0$, any connected component $\mathcal{D}_{k_1}^* \subsetneq \mathcal{D}$ will be an open interval with at least one extreme δ_0^* in $\mathring{\mathcal{D}} \setminus \mathcal{D}_{k_1}$, and therefore with $p_{k_1}(\delta_0^*) = 0$, (if the two extremes of $\mathcal{D}_{k_1}^*$ were the ones of $\mathcal{D}$, then $\mathcal{D}_{k_1}^* = \mathcal{D}$ but $\mathcal{D}_{k_1}^* \subsetneq \mathcal{D}$). By the continuity of $p_{k_1}(\delta)$, if a sequence δ_i in $\mathcal{D}_{k_1}^*$ converges to δ_0^*, then $p_{k_1}(\delta_i)$ converges to 0. ■

Remark 1. *When "*$\Pr(N_\delta = 1) > 0$ *for all* $N_\delta \in \mathcal{N}$*," as in the first statement in Theorem 1, and therefore when "*$\mathcal{D}_1 = \mathcal{D} = \mathcal{D}_{k_1}$*," one also has that* $\mathcal{D}_1$ *is connected and that* $\inf_{\delta\in\mathcal{D}_1} p_1(\delta) = 0$*, because* $N_\delta^{\circ m} = N_{\delta^{\circ m}} \in \mathcal{N}$ *for any* $\delta \in \mathcal{D}_1$ *and any* $m \in \mathbb{N}$*, and* $\Pr(N_\delta^{\circ m}) = p_1(\delta^{\circ m}) = p_1(\delta)^m$ *which tends to* 0 *when* m *tends to infinity.*

From now on, $\mathcal{D}^*_{k_1}$ will stand for a connected component of $\mathcal{D}_{k_1}$, that is, an open interval with δ_0^* as at least one of its endpoints. Different from Lemma 1, the next three lemmas relate to properties of all models "uniparametric and closed under pgf composition".

Lemma 2. *If* $\mathcal{N}$ *is "uniparametric and closed under pgf composition", then one can not have* $p_{k_1}(\delta) = C \neq 0$ *for* $\delta \in (\delta_1 - \varepsilon, \delta_1 + \varepsilon) \subset \mathcal{D}^*_{k_1}$ *with* $\varepsilon > 0$*.*

Proof: The result will be proven by *reductio ad absurdum*, checking that if $p_{k_1}(\delta) = C \neq 0$ in an interval in $\mathcal{D}^*_{k_1}$, the model can not be uniparametric.

If $p_k(\delta) = C_k \neq 0$ and so $p'_k(\delta) = 0$ for $\delta \in (\delta_1 - \varepsilon, \delta_1 + \varepsilon)$ and for all $k > k_1$, the parametrization δ would not be identifiable. Let $m > k_1$ be the first term such that there exists a $\delta'_1 \in (\delta_1 - \varepsilon, \delta_1 + \varepsilon)$ with $p'_m(\delta'_1) \neq 0$ and so by the continuity of the derivative such that $p_m(\delta)$ is strictly monotone in an environment $(\delta'_1 - \varepsilon', \delta'_1 + \varepsilon') \subset (\delta_1 - \varepsilon, \delta_1 + \varepsilon)$, and let δ'_1 and ε' be such that $p_k(\delta) = C_k$ in $(\delta'_1 - \varepsilon', \delta'_1 + \varepsilon')$ for all $k_1 \leq k < m$. (If $p'_m(\delta_1) \neq 0$, one can choose $\delta'_1 = \delta_1$). This means that in this interval one can parametrize the N_δ through $\theta = p_m(\delta)$. From now on, we relabel δ'_1 by δ_1, ε' by ε and the C of the statement of the lemma by C_{k_1}.

If d_1, d_2 are in $(\delta_1 - \varepsilon, \delta_1 + \varepsilon)$, then $N_{d_1} \circ N_{d_2}$ belongs to $\mathcal{N}$ and its pgf is:

$$h_{N_{d_1}} \circ h_{N_{d_2}}(t) = p_{k_1}(d_1)(h_{N_{d_2}}(t))^{k_1} + \cdots + p_m(d_1)(h_{N_{d_2}}(t))^m + \cdots$$

The first term of this pgf where $\theta_2 = p_m(d_2)$ appears is

$$p_{k_1}(d_1)(h_{N_{d_2}}(t))^{k_1} = p_{k_1}(d_1)(p_{k_1}(d_2)t^{k_1} + \cdots + p_m(d_2)t^m + \cdots)^{k_1} =$$

$$C_{k_1}(C_{k_1}t^{k_1} + \cdots + C_{m-1}t^{m-1} + \theta_2 t^m + \cdots + p_j(\theta_2)t^j + \cdots)^{k_1},$$

and it is under the form

$$k_1 C_{k_1}(C_{k_1}t^{k_1})^{k_1-1}\theta_2 t^m,$$

and therefore $\theta_2 = p_m(d_2)$ appears first in the coefficient of the term $t^{k_1^2+m-k_1}$ of the pgf of $N_{d_1} \circ N_{d_2}$, with a coefficient equal to $k_1 C^{k_1}\theta_2 + Q$, where Q is a constant term that depends on $p_k(d_1) = p_k(d_2) = C_k$ for $k_1 \leq k < m$. Therefore that first term only depends on θ_2.

Analogously, the first term of the pgf of $N_{d_1} \circ N_{d_2}$ where $\theta_1 = p_m(d_1)$ appears is $p_m(d_1)(h_{N_{d_2}}(t))^m$, and it is under the form $p_m(d_1)(p_{k_1}(d_2)t^{k_1})^m$, and therefore $\theta_1 = p_m(d_1)$ appears first in the coefficient of the term t^{mk_1} of that pgf, with a coefficient equal to $C^m\theta_1 + R(\theta_2)$, where $R(\theta_2)$ are terms that depend on $p_k(d_1) = C_k$ for $k_1 \leq k < m$ and on $\theta_2 = p_m(d_2)$.

Given that θ_1 and θ_2 are not related, one needs two parameters to describe $N_{d_1} \circ N_{d_2} \in \mathcal{N}$, and $\mathcal{N}$ can not be uniparametric, which proves the result. ■

Remark 2. *One consequence of Lemma 2 is that $p'_{k_1}(\delta)$ can not be zero in any interval, and therefore the number of critical points of $p_{k_1}(\delta)$ in $\mathcal{D}^*_{k_1}$, with $p'_{k_1}(\delta) = 0$, is either finite or countable.*

Lemma 3. *If $\mathcal{N}$ is "uniparametric and closed under pgf composition", then $p_{k_1}(\delta)$ is strictly monotone in $\mathcal{D}^*_{k_1}$, and therefore the N_δ with $\delta \in \mathcal{D}^*_{k_1}$ can be parametrized through $\theta = p_{k_1}(\delta)$.*

Proof: One assumes that $p_{k_1}(\delta)$ is continuously differentiable and so continuous, with $0 < p_{k_1}(\delta) \leq 1$ for all $\delta \in \mathcal{D}^*_{k_1}$. By Lemma 2 one knows that $p_{k_1}(\delta)$ is not constant in any subinterval of $\mathcal{D}^*_{k_1}$, and by Lemma 1 and Remark 1 one knows that one always has that $\inf_{\delta \in \mathcal{D}^*_{k_1}} p_1(\delta) = 0$.

To prove that $p_{k_1}(\delta)$ is strictly monotone in $\mathcal{D}^*_{k_1}$, one needs to prove that there are no local maxima or local minima in its interior, and no fluctuating critical points in its closure, where a fluctuating critical point δ_f is the limit of critical points such that $p_{k_1}(\delta)$ fluctuates when δ converges to δ_f. In our case though, it will be sufficient to check that there is no local maximum of $p_{k_1}(\delta)$ in the interior of $\mathcal{D}^*_{k_1}$, because:

- if there is no local maximum there can not be any fluctuating critical points δ_f, because δ_f would have to be the accumulation point of fluctuating critical points (the points where $p_{k_1}(\delta)$ fluctuates from increasing to decreasing can neither be local maxima nor local minima), and as a consequence the set of fluctuating critical points δ_f in $\bar{\mathcal{D}}^*_{k_1}$ would be a perfect set and the number of critical points in $\mathcal{D}^*_{k_1}$ would have a cardinality larger than countable, which is impossible by Remark 2,

- and if there is no local maximum in the interior of $\mathcal{D}^*_{k_1}$ and no fluctuating critical point in its closure, there can not exist an isolated minimum in there either, because if it did $p_{k_1}(\delta)$ would not be able to take values close enough to 0 near the boundary of $\mathcal{D}^*_{k_1}$.

That there is no local maximum is proven by *reductio ad absurdum*. Assume that $p_{k_1}(\delta)$ had a local maximum, δ_m, with $\delta_1 < \delta_m < \delta_2$ where $\delta_1, \delta_2 \in \mathcal{D}^*_{k_1}$, and by continuity of the derivative with $p'_{k_1}(\delta) \geq 0$ for $\delta \in (\delta_1 - \varepsilon, \delta_m)$ and with $p'_{k_1}(\delta) \leq 0$ for $\delta \in (\delta_m, \delta_2 + \varepsilon)$, and let $q = p_{k_1}(\delta_1) = p_{k_1}(\delta_2) < p_{k_1}(\delta_m) = q_m$. Because of the sign of the derivative, $p_{k_1}(\delta)$ would be increasing in $(\delta_1 - \varepsilon, \delta_m)$ and by Lemma 2 it would be strictly increasing there, and $p_{k_1}(\delta)$ would be strictly decreasing in $(\delta_m, \delta_2 + \varepsilon)$ for the same reasons. This would mean that, because of continuity, this distributions can be parametrized through $p_{k_1} = p_{k_1}(\delta)$ both in $[\delta_1, \delta_m]$ as well as in $[\delta_m, \delta_2]$.

Let $\delta, \delta^* \in \mathcal{D}_{k_1}$, and consider the family of distributions $N_{\tilde{\delta}} = N_\delta \circ N_{\delta^*} \in \mathcal{N}_{k_1} \circ \mathcal{N}_{k_1} \subset \mathcal{N}_{k_1^2} \subset \mathcal{N}^{\circ 2} \subset \mathcal{N}$, which is also uniparametric, closed under pgf composition, and with a $k_1^{'}$ equal to k_1^2, and lets denote $\tilde{\delta}(\delta, \delta^*)$ by $\delta \circ \delta^*$.

If $\delta, \delta^* \in \mathcal{D}^*_{k_1}$, then $\delta \circ \delta^* \in \mathcal{D}^*_{k_1^2}$, which is a connected subset of $\mathbb{R}$, and if $\delta, \delta^* \in [\delta_1, \delta_2] \subset \mathcal{D}^*_{k_1}$, then $\delta \circ \delta^* \in [\delta_1, \delta_2]^{\circ 2} \subset \mathcal{D}^*_{k_1^2}$. If $\delta \in [\delta_1, \delta_2]$, then $\{\delta \circ \delta\} = [\delta_1^{\circ 2}, \delta_2^{\circ 2}]$ is a connected subset of $[\delta_1, \delta_2]^{\circ 2}$, because one knows that if $\delta \neq \delta^*$ then $\delta^{\circ 2} \neq \delta^{*\circ 2}$, and that by continuity of the parametrization, if $\delta_j \in (\delta_1, \delta_2)$ converges to δ in $[\delta_1, \delta_2]$, then $\delta_j^{\circ 2}$ converges to $\delta^{\circ 2}$. The set $[\delta_1^{\circ 2}, \delta_2^{\circ 2}]$, (or the set $[\delta_2^{\circ 2}, \delta_1^{\circ 2}]$ if $\delta_2^{\circ 2} < \delta_1^{\circ 2}$), can be parametrized through $\delta^{\circ 2}$, and with that parametrization one has that $p_{k_1^2}(\delta^{\circ 2}) = p_{k_1}(\delta)p_{k_1}(\delta)^{k_1} = p_{k_1}(\delta)^{k_1+1} \in [q^{k_1+1}, q_m^{k_1+1}] \subset [0,1]$. In particular, $p_{k_1^2}(\delta_1^{\circ 2}) = p_{k_1^2}(\delta_2^{\circ 2}) = q^{k_1+1}$ and $p_{k_1^2}(\delta_m^{\circ 2}) = q_m^{k_1+1}$.

Consider now the set $\{\delta_1 \circ [\delta_1, \delta_2]\} \subset [\delta_1, \delta_2]^{\circ 2}$, with a first value $\delta_1^{\circ 2}$ that belongs to $[\delta_1^{\circ 2}, \delta_m^{\circ 2}]$. Given that the set $\{\delta_1 \circ [\delta_1, \delta_2]\}$ is connected and of dimension one, it has to be contained in $[\delta_1^{\circ 2}, \delta_m^{\circ 2}]$, because otherwise it would either contain $\delta_m^{\circ 2}$ or take values smaller than $\delta_1^{\circ 2}$, but $p_{k_1^2}(\delta_1 \circ [\delta_1, \delta_2]) \subset [p_{k_1^2}(\delta_1^{\circ 2}), p_{k_1^2}(\delta_m^{\circ 2}))$ and so its last value, $\delta_1 \circ \delta_2$, is in $[\delta_1^{\circ 2}, \delta_m^{\circ 2}]$; Given that $p_{k_1^2}(\delta_1 \circ \delta_2) = q^{k_1+1}$ and that $\delta_1^{\circ 2}$ is the unique point in $[\delta_1^{\circ 2}, \delta_m^{\circ 2}]$ such that $p_{k_1^2}(\tilde{\delta}) = p_{k_1^2}(\delta_1 \circ \delta_2) = q^{k_1+1}$, then $\delta_1 \circ \delta_2 = \delta_1 \circ \delta_1$. But this implies that $h_{N_{\delta_1}}(h_{N_{\delta_2}}(t)) = h_{N_{\delta_1}}(h_{N_{\delta_1}}(t))$, and so that $\delta_2 = \delta_1$, which is in contradiction with $\delta_1 < \delta_m < \delta_2$ and therefore there can not exist local maxima of $p_{k_1}(\delta)$ in the interior of $\mathcal{D}_{k_1}^*$.

As a consequence, $\theta = p_{k_1}(\delta)$ is strictly monotone in $\mathcal{D}_{k_1}^*$. ■

Lemma 4. *If $\mathcal{N}$ is "uniparametric and closed under pgf composition", then $k_1 = 1$.*

Proof: We prove that if $k_1 > 1$, then $\mathcal{N}$ can not be "uniparametric and closed under pgf composition" by *reductio ad absurdum*, in two steps.

In the first step we show that if $\mathcal{N}$ is "uniparametric and closed under pgf composition" and $k_1 > 1$, then $p_{k_1+j}(\theta) = \theta P_j(\theta)$ for all $j \geq 0$ and for every $\delta \in \mathcal{D}_{k_1}^*$, where $\theta = p_{k_1}(\delta)$ and $P_j(\theta)$ is a polynomial of θ.

In the second step we show that if $k_1 > 1$ and δ_i converges to δ_0 and so $p_{k_1}(\delta_i)$ converges to 0, then $p_{k_1+j}(\delta_i)$ converges to 0 for all $j \geq 0$, which given that $p_r(\delta_i) = 0$ for all $r < k_1$ implies that all the coefficients of $h_{\delta_0}(t)$ would be 0 and so $\delta_0 \notin \mathcal{D}$, but that would contradict Lemma 1.

1. Step 1: We prove that if $\mathcal{N}$ is "uniparametric and closed under pgf composition" and $k_1 > 1$, then $p_{k_1+j}(\theta) = \theta P_j(\theta)$, by induction.

   It holds for $j = 0$, because $p_{k_1}(\theta) = \theta P_0(\theta)$ with $P_0(\theta) = 1$. Lets assume that $p_{k_1+j}(\theta) = \theta P_j(\theta)$ for $j \leq m-1$, and check what happens for $j = m$.

   For any $\delta_1, \delta_2 \in \mathcal{D}_{k_1}^*$ one has that $N_{\delta_1} \circ N_{\delta_2} \in \mathcal{N}^{\circ 2} \subset \mathcal{N}_{k_1^2} \subset \mathcal{N}$, and

   $$h_{N_{\delta_1} \circ N_{\delta_2}} = \sum_{r=0}^{\infty} \left(p_{k_1+r}(\delta_1) \left(\sum_{j=0}^{\infty} p_{k_1+j}(\delta_2) t^{k_1+j} \right)^{k_1+r} \right)$$

   which by Lemma 3 can be parametritzed through $\tau = p_{k_1^2}(\delta) = p_{k_1}(\delta_1)p_{k_1}(\delta_2)^{k_1} = \theta_1 \theta_2^{k_1}$, and thus with $\theta_1 = \tau / \theta_2^{k_1}$. The first term in $h_{N_{\theta_1} \circ N_{\theta_2}}$ where $p_{k_1+m}(\theta_2)$ appears is:

   $$p_{k_1}(\delta_1) k_1 (p_{k_1}(\delta_2) t^{k_1})^{k_1-1} p_{k_1+m}(\delta_2) t^{k_1+m},$$

which is the term of order k_1^2+m, and its coefficient is of the form

$$p_{k_1^2+m}(\tau)=\sum_r p_{k_1+r}(\theta_1)\prod_{i=1}^{k_1+r}p_{k_1+j_i}(\theta_2),$$

where the sum is over all r and j_i such that $0\le r$, $0\le j_i$ and $0\le\sum_{i=1}^{k_1+r}j_i=m-k_1r$, and therefore it can be written as:

$$p_{k_1^2+m}(\tau)=k_1p_{k_1}(\theta_1)(p_{k_1}(\theta_2))^{k_1-1}p_{k_1+m}(\theta_2)+A,$$

where the first summand is the sum of the k_1 terms where $r=0$ and all $j_i=0$ except one j_i that is equal to m, and where A is the sum of the remaining terms, all with $j_i<m$. The first term where $p_{k_1+m}(\theta_1)$ appears is the one of order $k_1^2+k_1m$, larger than k_1^2+m because $k_1>1$, and therefore the $p_{k_1+j}(\theta_1)$ with $j\ge m$ do not appear in the above expression for $p_{k_1^2+m}(\tau)$.

Substituting $\theta_1=\tau/\theta_2^{k_1}$ and using the fact that $p_{k_1+j}(\theta)=\theta P_j(\theta)$ when $j<m$, the first sumand in the last expression for $p_{k_1^2+m}(\tau)$ becomes:

$$k_1\theta_1\theta_2^{k_1-1}p_{k_1+m}(\theta_2)=k_1\frac{\tau}{\theta_2}p_{k_1+m}(\theta_2),$$

and the second sumand in that expression becomes:

$$A=\sum_r p_{k_1+r}(\theta_1)\prod_{i=1}^{k_1+r}p_{k_1+j_i}(\theta_2)=\sum_r\theta_1P_r(\theta_1)\prod_{i=1}^{k_1+r}\theta_2P_{j_i}(\theta_2)=$$

$$\sum_r\tau P_r\left(\frac{\tau}{\theta_2^{k_1}}\right)\frac{\theta_2^{r+k_1}}{\theta_2^{k_1}}\prod_{i=1}^{k_1+r}P_{j_i}(\theta_2),$$

where the sum is over the set of r and j_i described above but excluding $j_i=m$. Hence, the coefficient of the k_1^2+m term of $h_{N_{\delta_1}\circ N_{\delta_2}}$ is a polynomial in τ of the form:

$$p_{k_1^2+m}(\tau)=\tau\left(\frac{k_1p_{k_1+m}(\theta_2)}{\theta_2}+P_{m(1)}(\theta_2)\right)+\sum_{i=2}^{c}\tau^iP_{m(i)}(\theta_2).$$

If $\mathcal{N}$ is uniparametric and closed under pgf composition, then all the coefficients of $h_{N_{\delta_1}\circ N_{\delta_2}}$ have to depend only on τ, and hence all the coefficients of this polynomial in τ need to be constant. In particular, for the coefficient of the first term in τ to be constant, equal to C_m, one needs that $k_1p_{k_1+m}(\theta_2)=\theta_2(C_m-P_{m(1)}(\theta_2))$, and therefore that $p_{k_1+m}(\theta)=\theta P_m(\theta)$.

2. Step 2: If $k_1>1$ and $\theta_i=p_{k_1}(\delta_i)$ converges to 0 when δ_i converges to δ_0, then it follows that $p_{k_1+m}(\delta_i)=p_{k_1}(\delta_i)P_m(p_{k_1}(\delta_i))$ will converge to 0 for all $m\ge0$, and by definition of k_1, $p_j(\delta_i)=0$ for all $j<k_1$. Therefore $h_{\delta_0}(t)=0$, which is not a pgf and so $\delta_0\notin\mathcal{D}$. By Lemma 1, if $k_1>1$ there is one such $\delta_0\in\mathring{\mathcal{D}}$ on the boundary of $\mathcal{D}_{k_1}^*$, but we just proved that $\delta_0\notin\mathring{\mathcal{D}}$, and therefore k_1 can not be larger than 1. ■

Proof of Theorem 1, in Section 6:

The fact that when $\mathcal{N}$ is "*uniparametric and closed under pgf composition*", then $p_1(\delta) = \Pr(N_\delta = 1) > 0$ for all $\delta \in \mathcal{D}$, follows from the fact that $k_1 = 1$ by Lemma 4, and that if there existed a $\delta_0 \in \mathring{\mathcal{D}}$ with $p_1(\delta_0) = 0$, then the model $\mathcal{N}_0 = \{N_\delta \circ N_{\delta_0} : N_\delta \in \mathcal{N}\}$ would also be "uniparametric and closed under pgf composition", because $(N_{\delta_1} \circ N_{\delta_0}) \circ (N_{\delta_2} \circ N_{\delta_0}) = (N_{\delta_1} \circ N_{\delta_0} \circ N_{\delta_2}) \circ N_{\delta_0} \in \mathcal{N}_0$, but $\mathcal{N}_0$ would have $k_1 > 1$ because $P(N_\delta \circ N_{\delta_0} = 1) = p_1(\delta_N) p_1(\delta_0) = 0$ for any $N_\delta \in \mathcal{N}$, in contradiction with Lemma 4.

From Lemma 3 it follows that $p_1(\delta)$ is strictly monotone in $\mathcal{D}$, and as a consequence one can parametrize $\mathcal{N}$ through $\theta = p_1(\delta)$. And from Remark 1 one knows that $\inf_{\delta\in\mathcal{D}} p_1(\delta) = 0$, hence the parameter space is $(0, p_{10}]$, where $p_{10} = \max_{\delta\in\mathcal{D}} p_1(\delta)$. Equivalently, one can parametrize $\mathcal{N}$ through $\eta = -\log p_1(\delta)$ with parameter space $[\eta_0, \infty)$, where $\eta_0 = -\log p_{10}$. ∎