SORT 49 (1) January-June 2025, 1-20 DOI:

Lattice structures for the stochastic comparison of call ratio backspread derivatives with an application

María Concepción López-Díaz[1], Miguel López-Díaz[2] and Sergio Martínez-Fernández[3]

Abstract

The comparison of investments in financial derivatives is an appealing topic in the optimization of resources. A relevant derivative is the call ratio backspread. Motivated by the need to compare investments in such derivatives, a new family of stochastic orders is introduced. That permits to reach decisions on the allocations of funds in those derivatives under general conditions and without assuming specific probability distributions of the asset prices. Characterizations of the orders are developed. Special emphasis is placed on the existence of infima and suprema in such dominance criteria, which leads to lattice structures on some special spaces and to the reduction of some optimization problems with stochastic dominance constraints. The method is illustrated with an application using real data from financial markets.

MSC: *60E15, 62P05.*

Keywords: *Call ratio backspread derivative, integrated survival function, lattice, stochastic order.*

1. Motivation of the study

The aim of this manuscript is to show how the theory of stochastic orders can be used for reaching decisions on the allocations of funds in call ratio backspread derivatives, entailing some advantages with respect to other methods.

[1] Departamento de Matemáticas, Universidad de Oviedo. C/ Leopoldo Calvo Sotelo 18. E-33007 Oviedo, Spain. cld@uniovi.es

[2] Departamento de Estadística e I.O. y D.M., Universidad de Oviedo. C/ Leopoldo Calvo Sotelo 18. E-33007 Oviedo, Spain. mld@uniovi.es

[3] Unidad de Auditoría de Capital & Impairments, Banco Sabadell. C/ Sena 12, P.I. Can Sant Joan. E-08174 Sant Cugat del Valles, Barcelona, Spain. martinezserg@bancsabadell.com
Received: June 2024.
Accepted: March 2025.

A financial derivative is a financial contract with a value derived from the future price of an underlying asset. Basically, it is an agreement between two parts, a buyer and a seller, that specifies conditions on the dates, resulting values, definitions of the underlying assets, the parties' contractual obligations, and the amount under which payments are to be made between the parties. The assets of derivatives can be of a quite different kind, such as goods and shares, but also indices such as rates of return, rates of interest or exchange rates.

A relevant derivative in financial markets is the (European) call option. That is an agreement that gives to the purchaser of the call option the right, but not the obligation, to buy an agreed quantity of an underlying asset at a specified exercise price, at an exercise date, while paying a premium for this right. If x is the unit price of the underlying asset on the due date, p is the unit exercise price and k is the unit premium, the benefit of the purchaser of the call option per unit of the asset is $(x-p)_+ - k$, where the subscript $+$ denotes the positive part of a real number. The unit profit of the seller of the call option is $k-(x-p)_+$.

Some financial derivatives are formed by means of other derivatives, like the call ratio backspread. Consider the common call ratio backspread, in which two call options are bought with unit exercise price p_2, and a call option is sold with unit exercise price p_1, where $p_1 < p_2$, all of them with the same asset and the same exercise date. From now on, we will refer to it as the call ratio backspread.

The unit benefit of a call ratio backspread is $k_1-(x-p_1)_+ + 2(x-p_2)_+ - 2k_2$, where x is the unit price of the asset at the exercise date, p_1 is the asset unit exercise price of the sale of the call option, k_1 its unit premium and p_2 and k_2 play the same role in the purchases of the call options.

In this manuscript, we propose a model to compare investments in call ratio backspread derivatives in terms of a family of stochastic orders, which does not need specific distributions of the asset prices and makes it possible to detect arbitrage options in financial markets, that is, detecting deals that would lead to a non-zero probability of future profit. Another advantage of the new method is that when the order is satisfied, the expected benefits are ordered whatever price p_1. Thus, an investor does not need to attain some particular values of p_1 to be able to compare investments and find opportunities. When the order is satisfied and the premiums do not follow the same arrangement for a particular value of p_1, there exist arbitrage opportunities. Moreover, we prove that there exist an infinum and a supremum of any two random variables with finite means with respect to any stochastic order of that family. The existence of a supremum and an infimum is useful in optimization problems with stochastic dominance constraints.

The reader is referred, for instance, to the books Dixit and Pindyck (1994), Cohen (2005) and Hull (2015) for an introduction to the field of financial derivatives, and to Müller and Stoyan (2002), Shaked and Shanthikumar (2007), Belzunce, Martínez-Riquelme and Mulero (2016) and Levy (2016) for a comprehensive introduction to the theory and applications of stochastic orderings. Some references which relate stochastic dominance criteria and arbitrage opportunities are Levy (2016), Jarrow (1986) and Ng, Wong and Xiao (2017), to the best of our knowledge, few manuscripts approach both topics. An article connecting stochastic orders and financial derivatives is López-Díaz, López-Díaz and Martínez-Fernández (2018).

As an application of the proposed model to compare investments in call ratio backspread derivatives, in the present manuscript we compare call ratio backspread derivatives whose assets are the weekly returns of Boeing and Procter & Gamble (P&G), companies in the Dow Jones Industrial Average Index. For that purpose, a result which permits to use statistical inference techniques to test conditions that lead to the call ratio backspread stochastic order is proved. As a consequence, we obtain that the expected benefit of a call ratio backspread derivative with asset the unit weekly revaluation of Boeing is greater (not lower) than the corresponding derivative with the asset unit weekly revaluation of P&G, whatever $p_1 < 1$. Then, if for some $p_1 < 1$, the premium of the derivative associated with Boeing is lower than the premium of P&G, an arbitrage opportunity exists for those derivatives. Moreover, in case of equality of premiums, an investor should choose the Boeing derivative instead of the P&G option.

The structure of the paper is as follows. Section 2 contains the preliminaries of the manuscript. In Section 3, we introduce the mathematical model to analyze the aforementioned problem in terms of a family of stochastic orders and we develop the main characterizations of those families. Section 4 is devoted to the analysis of the existence of infimum and supremum in such orderings. The application described above of the proposed method is developed in Section 5. To conclude, Section 6 contains some final comments and conclusions about the manuscript.

2. Preliminaries

In this section, preliminary concepts and notations are presented.

Throughout the paper, if $a \in \mathbb{R}$, a_+ will stand for $\max\{a,0\}$ and a_- for $\max\{-a,0\}$.

Given a random variable X, F_X will represent its distribution function, EX its expected value and P_X the probability induced by X. Moreover, $\overline{F}_X$ will denote the survival function of X.

The integrated survival function of a random variable X with finite mean is the mapping $\pi_X : \mathbb{R} \to \mathbb{R}$, with $\pi_X(t) = E(X-t)_+$ for any $t \in \mathbb{R}$. It is well-known that $\pi_X(t) = \int_{(t,+\infty)} \overline{F}_X(x)\,dx$.

A stochastic order is a pre-order relation on a set of probabilities. Basically, it aims to order probabilities in accordance with a criterion.

An integral stochastic order $\preceq$ is defined by the comparison of the integrals of real measurable mappings in a certain class. Namely, two probabilities P and Q on $(\mathbb{R}, \mathcal{B})$ ($\mathcal{B}$ denotes the usual Borel σ-field) satisfy $P \preceq Q$, when

$$\int_{\mathbb{R}} f\,dP \leq \int_{\mathbb{R}} f\,dQ$$

for any f in that class, such that the integrals exist. That set of mappings is said to be a generator of the order (see Müller (1997) for integral stochastic orders).

If $\preceq$ is a stochastic order on the probabilities on $(\mathbb{R},\mathcal{B})$, and X and Y are two random variables, $X \preceq Y$ will mean $P_X \preceq P_Y$.

The following integral stochastic orders will appear in the manuscript.

Let X and Y be random variables, then

i) X is said to be smaller than Y in the usual stochastic ordering ($X \preceq_{st} Y$) if $E(f(X)) \leq E(f(Y))$ for all increasing mappings $f : \mathbb{R} \to \mathbb{R}$ such that the above expectations exist, equivalently, $F_X \geq F_Y$,

ii) X is said to be smaller than Y in the increasing concave order ($X \preceq_{icv} Y$) if $E(f(X)) \leq E(f(Y))$ for all increasing concave mappings $f : \mathbb{R} \to \mathbb{R}$ such that the above expectations exist.

The notation $X \sim_{st} Y$ will mean that X and Y have the same distribution.

Given P a probability on $(\mathbb{R},\mathcal{B})$ and $T : \mathbb{R} \to \mathbb{R}$ a measurable mapping, $P \circ T^{-1}$ will denote the probability given by $P \circ T^{-1}(B) = P(T^{-1}(B))$ for any $B \in \mathcal{B}$.

If $A \subset \mathbb{R}$, I_A will stand for the indicator mapping of the set A.

3. The call ratio backspread stochastic orders

In this section, we introduce the new family of stochastic orders which arises from the aim to compare call ratio backspread derivatives, providing characterization results of those orders.

From now on, $f_{p_1,p_2,k}$ will stand for the mapping $f_{p_1,p_2,k} : \mathbb{R} \to \mathbb{R}$, with $f_{p_1,p_2,k}(x) = 2(x-p_2)_+ - (x-p_1)_+ + k$ for any $x \in \mathbb{R}$, with $p_1, p_2, k \in \mathbb{R}$, and $p_1 < p_2$. This mapping represents the unit benefit of a call ratio backspread at the expiration date with k equal to $k_1 - 2k_2$.

Given $p_2 \in \mathbb{R}$, we will denote by $\mathcal{F}^{p_2}$ the class of mappings $\mathcal{F}^{p_2} = \{f_{p_1,p_2,k} \mid p_1, k \in \mathbb{R}, p_1 < p_2\}$, that is, the family of mappings which represent the unit benefits of call ratio backspread derivatives where the unit exercise price of the call option purchases is a given value p_2.

Next, the model to compare call ratio backspread derivatives is introduced.

Definition 3.1. *Let X and Y be two random variables. It will be said that X is less than Y in the call ratio backspread stochastic order for the unit exercise price of the call option purchases p_2, if $E(f(X)) \leq E(f(Y))$ for any $f \in \mathcal{F}^{p_2}$ such that the above expectations exist. This relation will be denoted by $X \preceq_{crb}^{p_2} Y$.*

Consider two call ratio backsprad derivatives with common exercise prices and expiration dates. Assume now that their premiums are equal to some value k. Let X and Y stand for the random variables unit prices of the assets of those derivatives at the expiration date. The unit expected benefits of both financial derivatives are $E(f_{p_1,p_2,k}(X))$ and $E(f_{p_1,p_2,k}(Y))$, respectively. The relation $X \preceq_{crb}^{p_2} Y$ means that the expected benefit of the call ratio backspread derivative associated with Y is greater (not lower) than that of X, whatever unit exercise price p_1 of the call option sales and whatever premiums k.

Thus, if $X \preceq_{crb}^{p_2} Y$ is held and the real premium of the second derivative (that of Y) is lower than the premium of the first derivative, an option of arbitrage is being offered in financial markets.

Observe that the model does not assume specific probabilistic distributions of the prices of the underlying assets, such as Brownian movements.

Next, we state different characterization results of the call ratio backspread stochastic orders.

Given $p_2 \in \mathbb{R}$, let $\mathcal{F}_0^{p_2} = \{f_{p_1,p_2,p_2-p_1} \mid p_1 \in \mathbb{R},\, p_1 < p_2\}$. Notice that $\mathcal{F}_0^{p_2} \subset \mathcal{F}^{p_2}$. In fact, $\mathcal{F}_0^{p_2}$ is given by the mappings of the class $\mathcal{F}^{p_2}$ whose values at the point p_2 are equal to 0. Both $\mathcal{F}_0^{p_2}$ and $\mathcal{F}^{p_2}$ are generators of the stochastic order $\preceq_{crb}^{p_2}$. Observe that any mapping in $\mathcal{F}^{p_2}$ is a translation of a map in $\mathcal{F}_0^{p_2}$.

The following result says that the analysis of the family of call ratio backspread stochastic orders can be performed for the unit exercise price $p_2 = 0$.

Proposition 3.2. *Let X and Y be random variables. It holds that $X \preceq_{crb}^{p_2} Y$ if and only if $X - p_2 \preceq_{crb}^{0} Y - p_2$.*

Proof. Suppose that $X \preceq_{crb}^{p_2} Y$. Let $f \in \mathcal{F}^0$ such that $E(f(X-p_2))$ and $E(f(Y-p_2))$ exist. Let $T : \mathbb{R} \to \mathbb{R}$ with $T(x) = x - p_2$. We have that

$$\int_{\mathbb{R}} f(x)\, dP_{X-p_2} = \int_{\mathbb{R}} f(x)\, dP_X \circ T^{-1} = \int_{\mathbb{R}} f(T(x))\, dP_X = \int_{\mathbb{R}} f(x-p_2)\, dP_X.$$

Since $f \in \mathcal{F}^0$, $f = f_{p_1,0,k}$ for some $p_1 \in \mathbb{R}$ with $p_1 < 0$ and $k \in \mathbb{R}$. It can be seen that $f \circ T = f_{p_1+p_2,p_2,k}$, which belongs to $\mathcal{F}^{p_2}$. Thus,

$$\int_{\mathbb{R}} f(x-p_2)\, dP_X \leq \int_{\mathbb{R}} f(x-p_2)\, dP_Y = \int_{\mathbb{R}} f(x)\, dP_{Y-p_2},$$

which leads to $X - p_2 \preceq_{crb}^{0} Y - p_2$. The converse can be proved in a similar way. ■

All the results will be developed for the call ratio backspread stochastic order $\preceq_{crb}^{0}$, which we will refer to it as the call ratio backspread order. The counterpart for any unit exercise price p_2 can be immediately derived by Proposition 3.2.

Observe that the translations of the random variables in Proposition 3.2 can lead to variables and prices assuming negative values, even in the case that the original variables were positive.

Proposition 3.3. *Let X and Y be random variables with finite means. Then $X \preceq_{crb}^{0} Y$ if and only if*

$$-tF_X(t) + E(|X| I_{(t,\infty)}(X)) \leq -tF_Y(t) + E(|Y| I_{(t,\infty)}(Y)) \text{ for any } t < 0.$$

Proof. Assume that $X \preceq^0_{crb} Y$. Let $t<0$. Consider the mapping $f_{t,0,-t}$, which belongs to $\mathcal{F}^0_0$. The condition $X \preceq^0_{crb} Y$ implies that

$$\int_{\mathbb{R}} f_{t,0,-t}(x)\,dP_X \le \int_{\mathbb{R}} f_{t,0,-t}(x)\,dP_Y.$$

Now notice that

$$\int_{\mathbb{R}} f_{t,0,-t}(x)\,dP_X = -tP_X((-\infty,t]) + \int_{(t,0]} -x\,dP_X + \int_{(0,+\infty)} x\,dP_X$$

$$= -tF_X(t) + \int_{(t,+\infty)} |x|\,dP_X = -tF_X(t) + E(|X|I_{(t,\infty)}(X)),$$

which proves one implication.

Conversely, if

$$-tF_X(t) + E(|X|I_{(t,\infty)}(X)) \le -tF_Y(t) + E(|Y|I_{(t,\infty)}(Y))$$

for any $t<0$, then $E(f_{t,0,-t}(X)) \le E(f_{t,0,-t}(Y))$ for any $t<0$. Notice that the class of mappings $\{f_{t,0,-t} \mid t<0\}$ is $\mathcal{F}^0_0$, thus, $X \preceq^0_{crb} Y$. ■

The following result provides a characterization of $\preceq^0_{crb}$ in terms of integrated survival functions. It will be key to prove the existence of infimum and supremum in the order.

Proposition 3.4. *Let X and Y be random variables with finite means. We have that $X \preceq^0_{crb} Y$ if and only if*

$$-\pi_X(t) + 2\pi_X(0) \le -\pi_Y(t) + 2\pi_Y(0) \ \textit{for any}\ t<0.$$

Proof. Notice that $X \preceq^0_{crb} Y$ holds if and only if $E(f(X)) \le E(f(Y))$ for any $f \in \mathcal{F}^0_0$, that is, if and only if $E(f_{p_1,0,-p_1}(X)) \le E(f_{p_1,0,-p_1}(Y))$ for any $p_1<0$.

Observe that $E(f_{p_1,0,-p_1}(X)) = E(2X_+ - (X-p_1)_+ - p_1)$, and so, $X \preceq^0_{crb} Y$ is equivalent to $2EX_+ - E(X-p_1)_+ \le 2EY_+ - E(Y-p_1)_+$ for any $p_1<0$, that is, $2\pi_X(0) - \pi_X(t) \le 2\pi_Y(0) - \pi_Y(t)$ for any $t<0$. ■

Some consequences of the preceding results are developed below.

Proposition 3.5. *Let X and Y be random variables with finite means. If $X \preceq^0_{crb} Y$ and $Y \preceq^0_{crb} X$, then $X_- \sim_{st} Y_-$.*

Proof. By Proposition 3.4, $X \preceq^0_{crb} Y$ and $Y \preceq^0_{crb} X$ are equivalent to $-\pi_X(t)+2\pi_X(0) = -\pi_Y(t)+2\pi_Y(0)$ for any $t<0$.

By the Second Fundamental Theorem of Calculus, $\overline{F}_X = \overline{F}_Y$ almost everywhere in $(-\infty, 0)$. Since distribution functions are right continuous, $\overline{F}_X(t) = \overline{F}_Y(t)$ for any $t < 0$, that is, $\overline{F}_X(-t) = \overline{F}_Y(-t)$ for any $t > 0$.

Therefore, $(1 - F_X(-t))I_{(0,+\infty)}(t) = (1 - F_Y(-t))I_{(0,+\infty)}(t)$ for any $t > 0$, hence $(1 - F_X(-t^-))I_{(0,+\infty)}(t) = (1 - F_X(-t^-))I_{(0,+\infty)}(t)$ for any $t > 0$. This is the same as $F_{X_-}(t) = F_{Y_-}(t)$ for any $t > 0$, thus, $X_- \sim_{st} Y_-$. ■

Corollary 3.6. *The relation $\preceq^0_{crb}$ is a partial order on the set of a.s. negative random variables with finite means, where equality is in distribution.*

Proof. The reflexive property is obvious. Transitivity follows from Proposition 3.4. The anti-symmetric property is a consequence of Proposition 3.5. ■

The order $\preceq^0_{crb}$ is not a partial order on the set of random variables with finite mean, but a pre-order. Consider the random variables X and Y with $P(X = 0) = P(X = 2) = 1/2$ and $P(Y = 1) = 1$. It holds that $X \preceq^0_{crb} Y$ and $Y \preceq^0_{crb} X$, but $X \sim_{st} Y$ is false.

4. Lattice structures

Throughout this section, we will prove that there exist an infinum and a supremum of any two random variables with finite means with respect to any call ratio backspread stochastic order. This permits to construct lattice structures on special partially ordered sets.

We prove the following result on integrated survival functions to analyze the case of the infimum.

Proposition 4.1. *Let X and Y be random variables with finite means. Let $h : \mathbb{R} \to \mathbb{R}$ with*

$$h(t) = \max\{\pi_X(t) - 2\pi_X(0), \pi_Y(t) - 2\pi_Y(0)\} + 2\min\{\pi_X(0), \pi_Y(0)\}$$

for any $t \in \mathbb{R}$. The mapping h is an integrated survival function of a random variable with finite mean.

Proof. Notice that the mapping h is well-defined since X and Y have finite means.

In accordance with Theorem 1.5.10 in Müller and Stoyan (2002), the mapping h is an integrated survival function of a random variable with finite mean if and only if : $i)$ h is decreasing, $ii)$ h is convex, $iii)$ $\lim_{t\to+\infty} h(t) = 0$, and $iv)$ $\lim_{t\to-\infty} h(t) + t \in \mathbb{R}$.

In relation to $i)$, since π_X and π_Y are integrated survival functions, both are decreasing, and so is h.

Regarding the second condition, we should prove that for any $t_1, t_2 \in \mathbb{R}$ and any $\lambda \in [0,1]$, it holds that $h(\lambda t_1 + (1-\lambda)t_2) \leq \lambda h(t_1) + (1-\lambda)h(t_2)$.

Notice that π_X and π_Y are convex since they are integrated survival functions. Thus,

$$\pi_X(\lambda t_1 + (1-\lambda)t_2) - 2\pi_X(0)$$

$$\leq \lambda \pi_X(t_1) + (1-\lambda)\pi_X(t_2) - 2(\lambda \pi_X(0) + (1-\lambda)\pi_X(0))$$
$$= \lambda(\pi_X(t_1) - 2\pi_X(0)) + (1-\lambda)(\pi_X(t_2) - 2\pi_X(0)),$$

and the same inequality is satisfied with Y. Therefore,

$$\max\{\pi_X(\lambda t_1 + (1-\lambda)t_2) - 2\pi_X(0),\, \pi_Y(\lambda t_1 + (1-\lambda)t_2) - 2\pi_Y(0)\}$$

$$\leq \max\{\lambda(\pi_X(t_1) - 2\pi_X(0)) + (1-\lambda)(\pi_X(t_2) - 2\pi_X(0)),$$
$$\lambda(\pi_Y(t_1) - 2\pi_Y(0)) + (1-\lambda)(\pi_Y(t_2) - 2\pi_Y(0))\}$$
$$\leq \lambda \max\{\pi_X(t_1) - 2\pi_X(0), \pi_Y(t_1) - 2\pi_Y(0)\}$$
$$+(1-\lambda)\max\{\pi_X(t_2) - 2\pi_X(0), \pi_Y(t_2) - 2\pi_Y(0)\},$$

which leads to the convexity of h.

With respect to *iii*), $\lim_{t\to+\infty} \pi_X(t) = 0$ and $\lim_{t\to+\infty} \pi_Y(t) = 0$ since π_X and π_Y are integrated survival functions of random variables with finite means, therefore,

$$\lim_{t\to+\infty} h(t) = \max\{-2\pi_X(0), -2\pi_Y(0)\} + 2\min\{\pi_X(0), \pi_Y(0)\} = 0.$$

In relation to *iv*), X and Y have finite means, hence $\lim_{t\to-\infty} \pi_X(t) + t = EX$ and $\lim_{t\to-\infty} \pi_Y(t) + t = EY$. Thus,

$$\lim_{t\to-\infty} h(t) + t$$
$$= \lim_{t\to-\infty} \max\{\pi_X(t) - 2\pi_X(0),\, \pi_Y(t) - 2\pi_Y(0)\} + 2\min\{\pi_X(0), \pi_Y(0)\} + t$$
$$= \lim_{t\to-\infty} \max\{\pi_X(t) + t - 2\pi_X(0),\, \pi_Y(t) + t - 2\pi_Y(0)\} + 2\min\{\pi_X(0), \pi_Y(0)\}$$
$$= \max\{EX - 2\pi_X(0), EY - 2\pi_Y(0)\} + 2\min\{\pi_X(0), \pi_Y(0)\} \in \mathbb{R}.$$

Therefore, h is an integrated survival function of a random variable with finite mean.

■

If W is a random variable such that $\pi_W = h$, in accordance with Theorem 1.5.10 in Müller and Stoyan (2002), it holds that $EW = \lim_{t\to-\infty} h(t) + t$. Thus, $EW = \max\{EX - 2\pi_X(0), EY - 2\pi_Y(0)\} + 2\min\{\pi_X(0), \pi_Y(0)\}$.

Proposition 4.2. *Let X and Y be random variables with finite means. Let $h : \mathbb{R} \to \mathbb{R}$, with*

$$h(t) = \max\{\pi_X(t) - 2\pi_X(0),\, \pi_Y(t) - 2\pi_Y(0)\} + 2\min\{\pi_X(0), \pi_Y(0)\}$$

for any $t \in \mathbb{R}$. Let W be a random variable such that $\pi_W = h$. Then, W is an infimum of X and Y in the stochastic order $\preceq^0_{crb}$.

Proof. Proposition 4.1 guarantees the existence of a random variable W with finite mean in the conditions of the statement. Now, notice that

$$h(0) = \max\{-\pi_X(0), -\pi_Y(0)\} + 2\min\{\pi_X(0), \pi_Y(0)\} = \min\{\pi_X(0), \pi_Y(0)\}.$$

Therefore, we have that

$$\begin{aligned} -h(t) + 2h(0) = &-\max\{\pi_X(t) - 2\pi_X(0),\, \pi_Y(t) - 2\pi_Y(0)\} \\ &-2\min\{\pi_X(0), \pi_Y(0)\} + 2\min\{\pi_X(0), \pi_Y(0)\} \\ &= -\max\{\pi_X(t) - 2\pi_X(0),\, \pi_Y(t) - 2\pi_Y(0)\} \\ &= \min\{-\pi_X(t) + 2\pi_X(0),\, -\pi_Y(t) + 2\pi_Y(0)\}. \end{aligned}$$

Thus,

$$-h(t) + 2h(0) \leq -\pi_X(t) + 2\pi_X(0) \;\text{ and }\; -h(t) + 2h(0) \leq -\pi_Y(t) + 2\pi_Y(0)$$

for any $t < 0$. By Proposition 3.4, $W \preceq^0_{crb} X$ and $W \preceq^0_{crb} Y$.

Let Z be a random variable with finite mean such that $Z \preceq^0_{crb} X$ and $Z \preceq^0_{crb} Y$. In accordance with Proposition 3.4,

$$-\pi_Z(t) + 2\pi_Z(0) \leq -\pi_X(t) + 2\pi_X(0) \text{ and } -\pi_Z(t) + 2\pi_Z(0) \leq -\pi_Y(t) + 2\pi_Y(0)$$

for all $t < 0$. Thus, $-\pi_Z(t) + 2\pi_Z(0) \leq -h(t) + 2h(0)$ for all $t < 0$, equivalently, $Z \preceq^0_{crb} W$, which proves the result. ■

Proposition 4.3. *Let X and Y be random variables with finite means. Let Z be an infimum of X and Y in the order $\preceq^0_{crb}$. Then,*

i) $EZ_+ = \min\{EX_+, EY_+\}$,

ii) $EZ = \max\{EX - 2EX_+, EY - 2EY_+\} + 2\min\{EX_+, EY_+\}$.

Proof. Let W be the infimum of X and Y in $\preceq^0_{crb}$ given in Proposition 4.2. Since Z and W are infima, then $W \preceq^0_{crb} Z$ and $Z \preceq^0_{crb} W$, that is, $-\pi_Z(t)+2\pi_Z(0) = -\pi_W(t)+2\pi_W(0)$ for any $t<0$.

The Monotone Convergence Theorem implies that $\pi_Z(0) = \pi_W(0)$, which is equal to $\min\{EX_+, EY_+\}$, and so, we derive *i*).

On the other hand, we have obtained that $\pi_Z(t) = \pi_W(t)$ for any $t<0$, and so,

$$EZ = \lim_{t\to-\infty} \pi_Z(t)+t = \lim_{t\to-\infty} \pi_W(t)+t$$

$$= \max\{EX-2EX_+, EY-2EY_+\}+2\min\{EX_+, EY_+\},$$

which concludes the proof. ∎

Proposition 4.4. *Let X and Y be random variables with finite means such that $X \le 0$ a.s. Then, the infimum of X and Y with respect to the order $\preceq^0_{crb}$ is unique in distribution.*

Proof. Let us suppose that W_1 and W_2 are two infima of X and Y. Since $W_i \preceq^0_{crb} X$, we have that $E(f(W_i)) \le E(f(X))$ for any $f \in \mathcal{F}^0_0$ and any $i \in \{1,2\}$.

Take the sequence $\{f_{-\frac{1}{n},0,\frac{1}{n}}\}_n \subset \mathcal{F}^0_0$, which is decreasing and whose pointwise convergence is the mapping $g(x) = x_+$ for any $x \in \mathbb{R}$.

The Monotone Convergence Theorem implies that $EW_{i_+} \le EX_+ = 0$. Therefore, $W_{i_+} = 0$ *a.s.* and so $W_i \le 0$ *a.s.* for any $i \in \{1,2\}$.

Since W_1 and W_2 are infima of X and Y, we obtain that $W_1 \preceq^0_{crb} W_2$ and $W_2 \preceq^0_{crb} W_1$. Corollary 3.6 leads to $W_1 \sim_{st} W_2$. ∎

Next we analyze the case of the supremum of two random variables in the call ratio backspread stochastic order.

Let I be an interval of $\mathbb{R}$ with non-empty interior. Let $f : I \to \mathbb{R}$ be a mapping. We will denote by $vex(f)$ the mapping $vex(f) : I \to \mathbb{R}$, with

$$vex(f)(t) = \sup\{g(t) \mid g \text{ is convex and } g(x) \le f(x) \text{ for all } x \in I\}$$

for any $t \in I$. This mapping is usually known as the convex hull operator, or the greatest convex minorant.

Proposition 4.5. *Let X and Y be random variables with finite means. Then, there exists a random variable W with finite mean which is a supremum of X and Y in the stochastic order $\preceq^0_{crb}$.*

Proof. Let us consider the mapping $l : (-\infty, 0] \to \mathbb{R}$, with

$$l(t) = \min\{\pi_X(t)-2EX_+,\ \pi_Y(t)-2EY_+\}+2\max\{EX_+, EY_+\}$$

for any $t \leq 0$. It holds that

$$l(0) = \min\{\,-EX_+,\,-EY_+\,\} + 2\max\{EX_+, EY_+\} = \max\{EX_+, EY_+\} \geq 0.$$

Define $h : (-\infty, 0] \to \mathbb{R}$, with $h = vex(l)$.

Observe that h is decreasing. Notice that π_X and π_Y are decreasing since they are integrated survival functions, therefore l is decreasing and so is h.

Trivially h is convex.

Let us see that $\lim_{t\to-\infty} h(t) + t \in \mathbb{R}$. Notice that the mapping $t \to h(t) + t$ is convex, which guarantees the existence of that limit.

We have that for any $t \leq 0$ it holds that $h(t) + t \leq l(t) + t$. Now

$$\begin{aligned}\lim_{t\to-\infty} l(t) + t = \lim_{t\to-\infty} \min\{\,\pi_X(t) - 2EX_+,\, \pi_Y(t) - 2EY_+\,\} \\ +2\max\{EX_+, EY_+\} + t \\ = \lim_{t\to-\infty} \min\{\,\pi_X(t) + t - 2EX_+,\, \pi_Y(t) + t - 2EY_+\,\} + 2\max\{EX_+, EY_+\} \\ = \min\{\,EX - 2EX_+,\, EY - 2EY_+\,\} + 2\max\{EX_+, EY_+\} \in \mathbb{R}\end{aligned}$$

since X and Y have finite means.

Let $\widetilde{l} : \mathbb{R} \to \mathbb{R}$ given by $\widetilde{l}(t) = \min\{\,\pi_X(t), \pi_Y(t)\,\}$ for all $t \in \mathbb{R}$. Define the mapping $\widetilde{h} = vex(\widetilde{l})$.

Clearly $\widetilde{l}(t) \leq l(t)$ when $t \in (-\infty, 0]$. As a consequence $\widetilde{h}(t) \leq h(t)$ for any $t \in (-\infty, 0]$.

In accordance with Müller and Scarsini (2006), the function $\widetilde{h}$ is the integrated survival function of a random variable with finite expectation. Thus, $\lim_{t\to-\infty} \widetilde{h}(t) + t \in \mathbb{R}$.

Since $\lim_{t\to-\infty} \widetilde{h}(t) + t \leq \lim_{t\to-\infty} h(t) + t \leq \lim_{t\to-\infty} l(t) + t$, we conclude that $\lim_{t\to-\infty} h(t) + t \in \mathbb{R}$.

Consider now any mapping $\widehat{h} : \mathbb{R} \to \mathbb{R}$ with $\widehat{h}(t) = h(t)$ for any $t \leq 0$ such that $\widehat{h}$ is continuous, convex, decreasing and with $\lim_{t\to+\infty} \widehat{h}(t) = 0$. Thus, $\widehat{h}$ is an integrated survival function of a random variable with finite mean.

The existence of at least one of such mappings can be guaranteed as follows.

Notice that $l(0) = \max\{EX_+, EY_+\}$. Since the constant mapping $\max\{EX_+, EY_+\}$ is convex, and $l(t) \geq l(0)$ for any $t < 0$, it holds that $h(0) = l(0) = \max\{EX_+, EY_+\} \geq 0$.

If $h(0) = 0$ the extension is trivial by taking $\widehat{h}(t) = 0$ for all $t \geq 0$.

Let $h(0) > 0$. Since h is convex, there exists $h'_-(0)$, the left derivative of h at the point 0 (see, for instance, Roberts and Varberg (1973)). Moreover, $h'_-(0) \leq 0$ due to the decreasing of h.

If $h'_{-}(0) < 0$, consider the mapping $g : \mathbb{R} \to \mathbb{R}$ with $g(t) = h(0) + h'_{-}(0)t$ for all $t \geq 0$. This mapping is continuous, convex, strictly decreasing, cuts the x-axis, is tangent to h at the point 0 and $g(0) = h(0)$.

Thus, it is sufficient to take

$$\widehat{h}(t) = \begin{cases} h(t) & \text{if } t \leq 0, \\ g(t) & \text{if } t \in (0, -\frac{h(0)}{h'_{-}(0)}], \\ 0 & \text{if } t > -\frac{h(0)}{h'_{-}(0)}. \end{cases}$$

Let us see that $h'_{-}(0) = 0$ is not possible.

The condition $h(0) > 0$ implies that $\pi_X(0) > 0$ or $\pi_Y(0) > 0$. Let us suppose that $\pi_X(0) \geq \pi_Y(0)$ and so $0 < \pi_X(0)$.

Firstly, consider the case $0 < \pi_Y(0) < \pi_X(0)$.

Since $\pi_X(0) > 0$, π_X is convex and $\lim_{t \to +\infty} \pi_X(t) = 0$, we obtain that π_X is strictly decreasing (when different from 0) and so $\pi'_{X-}(0) < 0$ (left derivative of π_X at the point 0). Moreover,

$$l(t) = \min\{\, \pi_X(t), \pi_Y(t) + 2(EX_+ - EY_+)\,\}$$

for any $t \leq 0$.

Let $\widetilde{t} \leq 0$ such that $\pi'_{Y-}(\widetilde{t}) < 0$. Such a point exists since π_Y is an integrated survival function. Take $\alpha < 0$ satisfying that

i) $\pi'_{X-}(0) \leq \alpha$,

ii) $\pi_X(0) + \alpha\widetilde{t} < \pi_Y(0) + 2(\pi_X(0) - \pi_Y(0))$, and

iii) $\pi'_{Y-}(\widetilde{t}) \leq \alpha$.

Such a value α exists since $\pi_X(0) < \pi_Y(0) + 2(\pi_X(0) - \pi_Y(0))$.

Define $g : (-\infty, 0] \to \mathbb{R}$ with $g(t) = \pi_X(0) + \alpha t$ for any $t \leq 0$.

Let us see that $g(t) \leq l(t)$ for any $t \leq 0$.

By condition *i*), $g(t) \leq \pi_X(0) + \pi'_{X-}(0)t \leq \pi_X(t)$ for any $t \leq 0$.

On the other hand, if $t \in [\widetilde{t}, 0)$, condition *ii*) leads to $g(t) \leq \pi_Y(0) + 2(\pi_X(0) - \pi_Y(0)) \leq \pi_Y(t) + 2(\pi_X(0) - \pi_Y(0))$ for any $t \leq 0$.

Moreover, if $t \in (-\infty, \widetilde{t})$, conditions *ii*) and *iii*) imply that

$$g(t) = \pi_X(0) + \alpha\widetilde{t} + \alpha(t - \widetilde{t}) \leq \pi_Y(0) + 2(\pi_X(0) - \pi_Y(0)) + \alpha(t - \widetilde{t})$$

$$\leq \pi_Y(\widetilde{t}) + 2(\pi_X(0) - \pi_Y(0)) + \pi'_{Y-}(\widetilde{t})(t - \widetilde{t}) \leq \pi_Y(t) + 2(\pi_X(0) - \pi_Y(0)).$$

Therefore, $g(t) \leq \min\{\pi_X(t), \pi_Y(t) + 2(\pi_X(0) - \pi_Y(0))\} = l(t)$ for any $t \leq 0$. Since g is convex, $g(t) \leq h(t)$ for any $t \leq 0$. Recall that $g(0) = h(0)$. Thus,

$$0 > \alpha = \lim_{x \to 0^-} \frac{g(x) - g(0)}{x} \geq \lim_{x \to 0^-} \frac{h(x) - h(0)}{x} = h'_-(0).$$

Therefore, $h'_-(0) < 0$.

The case $0 < \pi_X(0) = \pi_Y(0)$ is immediate since $l = \min\{\pi_X, \pi_Y\}$. In this case $\widehat{h} = vex(l)$, where l is defined on the whole real line, satisfies the required conditions.

As a consequence, we have that the mapping $\widehat{h}$ is an integrated survival function of a random variable with finite mean.

Let W be a random variable whose integrated survival function fulfills $\pi_W = \widehat{h}$.

Let us see that W is a supremum of X and Y in the order $\preceq^0_{crb}$.

Let $t < 0$, then

$$\begin{gathered} \pi_W(t) - 2\pi_W(0) \\ \leq \min\{\pi_X(t) - 2EX_+, \pi_Y(t) - 2EY_+\} + 2\max\{EX_+, EY_+\} - 2\pi_W(0) \\ \leq \pi_X(t) - 2EX_+ = \pi_X(t) - 2\pi_X(0). \end{gathered}$$

Applying Proposition 3.4, we deduce that $X \preceq^0_{crb} W$. In the same way we obtain that $Y \preceq^0_{crb} W$.

Let Z be a random variable with $X \preceq^0_{crb} Z$ and $Y \preceq^0_{crb} Z$. For any $t < 0$

$$\pi_Z(t) - 2\pi_Z(0) \leq \pi_X(t) - 2\pi_X(0) \quad \text{and} \quad \pi_Z(t) - 2\pi_Z(0) \leq \pi_Y(t) - 2\pi_Y(0).$$

Thus, for any $t < 0$

$$\pi_Z(t) - 2\pi_Z(0) + 2\max\{EX_+, EY_+\} \leq l(t).$$

Notice that $\pi_Z(t) - 2\pi_Z(0) + 2\max\{EX_+, EY_+\}$ is convex since π_Z is convex, which implies that for any $t < 0$

$$\pi_Z(t) - 2\pi_Z(0) + 2\max\{EX_+, EY_+\} \leq \widehat{h}(t),$$

which is the same as $\pi_Z(t) - 2\pi_Z(0) \leq \widehat{h}(t) - 2\widehat{h}(0)$ for any $t < 0$, equivalently $\pi_Z(t) - 2\pi_Z(0) \leq \pi_W(t) - 2\pi_W(0)$ for any $t < 0$, that is, $W \preceq^0_{crb} Z$. This concludes the proof of the existence of a supremum in the stochastic order $\preceq^0_{crb}$. ■

Proposition 4.6. *Let X and Y be random variables with finite means. Let Z be a supremum of X and Y in the order $\preceq^0_{crb}$. Then, we have that $EZ_+ = \max\{EX_+, EY_+\}$.*

Proof. Let W be a random variable which is the supremum of X and Y given in Proposition 4.5. Thus, $\pi_W(0) = h(0) = \max\{EX_+, EY_+\}$.

Let Z be another supremum of X and Y in the stochastic order $\preceq^0_{crb}$. Thus, we have that $Z \preceq^0_{crb} W$ and $W \preceq^0_{crb} Z$. By Proposition 3.4 we obtain that $-\pi_Z(t) + 2\pi_Z(0) = -\pi_W(t) + 2\pi_W(0)$ for any $t < 0$. The Monotone Convergence Theorem implies that $\pi_W(0) = \pi_Z(0)$, and so we obtain the result. ■

Proposition 4.7. *Let X and Y be random variables with finite means such that $X \leq 0$ and $Y \leq 0$ a.s. Then, the supremum of X and Y with respect to the order $\preceq^0_{crb}$ is unique in distribution.*

Proof. Let W and Z be two suprema of X and Y in the order $\preceq^0_{crb}$, thus $W \preceq^0_{crb} Z$ and $Z \preceq^0_{crb} W$.

Since $X \leq 0$ and $Y \leq 0$ $a.s.$, it holds that $EX_+ = EY_+ = 0$. Applying Proposition 4.6, we obtain that $\pi_W(0) = 0$ and $\pi_Z(0) = 0$. That is, $EW_+ = 0 = EZ_+$, and so $W \leq 0$ and $Z \leq 0$ $a.s.$

Applying Corollary 3.6 we conclude that $Z \sim_{st} W$. ■

Under the assumption of no arbitrage opportunities, the supremum of two variables corresponds to the price of an asset of the call ratio backspread derivative with greater expected benefit (not lower) than those of the variables and with the smallest possible premium. In a similar way, the infimum is a distribution of the price of an asset of the best call ratio backspread which is cheaper (not more expensive) than those of the variables.

The existence of a supremum and an infimum is useful in optimization problems with stochastic dominance constraints (see, for instance, Dentcheva, Lai and Ruszczyński (2004), Dentcheva and Martínez (2012), Dentcheva and Wolfhagen (2016), Singh and Selvamuthu (2017), Consigli, Dentcheva and Maggioni (2021) and the references therein for stochastic dominance constraints and Müller and Scarsini (2006) for lattice of stochastic orders). In the problem

$$\begin{aligned} &\text{maximize} \quad h(X) \\ &\text{subject to} \quad X \preceq^0_{crb} W_i, \;\; i = 1, \ldots, m, \\ &X \in \mathcal{C} \end{aligned}$$

where $\mathcal{C}$ is a set of random variables, $h : \mathcal{C} \to \mathbb{R}$ is a real valued functional, and W_i, with $i = 1, \ldots, m$, are random variables, the stochastic constraint is equivalent to $X \preceq^0_{crb} \inf\{W_i,\ i = 1, \ldots, m\}$, where the infimum is in the stochastic order $\preceq^0_{crb}$. Thus, there is only one stochastic constraint instead of m. In a similar way, if the stochastic constraints are subject to $W_i \preceq^0_{crb} X,\ i = 1, \ldots, m$, this is equivalent to $\sup\{W_i,\ i = 1, \ldots, m\} \preceq^0_{crb} X$, the supremum being in the order $\preceq^0_{crb}$.

The order $\preceq^0_{crb}$ is not a partial order but a pre-order and lattice structures are defined on partially ordered sets. Let $\sim^0_{crb}$ be the equivalence relation given by $\preceq^0_{crb}$ in the usual way. The following result follows from Proposition 4.2 and Proposition 4.5.

Proposition 4.8. *Let $\mathcal{M}^1$ be the set of random variables with finite mean and $\mathcal{M}^1/\sim^0_{crb}$ be the set of equivalence classes in $\mathcal{M}^1$ with respect to $\sim^0_{crb}$. Let $\tilde{\preceq}^0_{crb}$ be the relation on $\mathcal{M}^1/\sim^0_{crb}$ given by $[X]\,\tilde{\preceq}^0_{crb}[Y]$ when $X \preceq^0_{crb} Y$. The set of equivalence classes $\mathcal{M}^1/\sim^0_{crb}$ endowed with $\tilde{\preceq}^0_{crb}$ is a lattice.*

An equivalence class is made up of random variables of unit prices of assets on the expiration date of call ratio backspread derivatives whose expected benefits are the same. If $X \in [Y]$, $X + p_2 \sim^{p_2}_{crb} Y + p_2$, for any $p_2 \in \mathbb{R}$, that is, $E(f_{p_1,p_2,k}(X + p_2)) = E(f_{p_1,p_2,k}(Y + p_2))$ whatever $p_1 < p_2$ and $k \in \mathbb{R}$. An equivalence class can be interpreted as those assets of call ratio backspread derivatives whose premiums should be equal if there were not options of arbitrage. If elements of an equivalence class can be bought with different premiums, opportunity of arbitrage are being offered in financial markets.

5. An application of the method

This section illustrates the method developed for the analysis of call ratio backspread derivatives. We will compare call ratio backspread derivatives whose assets are the weekly returns of Boeing and Procter & Gamble (P&G), companies in the Dow Jones Industrial Average Index.

Let x_t stand for the weekly close price of a market value at week t. Notice that $\frac{x_t}{x_{t-1}}$ is the price at the end of week t of a monetary unit invested in such a value at the end of week $t-1$. The weekly return is defined as $\frac{x_t}{x_{t-1}} - 1$, that is, the interest rate during the corresponding week.

We will consider the share prices of Boeing and P&G during the period 2019-23. The data of the prices are public and were taken from https://es.finance.yahoo.com/

In Figure 1, we have depicted the daily evolution of such prices during the above period. Figure 2 shows the evolution of the weekly returns.

Let X be the random variable weekly return of P&G and let Y stand for the variable associated with the weekly returns of Boeing.

To show empirical evidence of an ordering between the corresponding call ratio backspread derivatives, we will make use of Proposition 3.4, which reads that $X \preceq^0_{crb} Y$ if and only if $-\pi_X(t) + 2\pi_X(0) \leq -\pi_Y(t) + 2\pi_Y(0)$ for any $t < 0$.

We have depicted the empirical version of the above quantities, that is, $-\widetilde{\pi}_X(t) + 2\widetilde{\pi}_X(0)$ and $-\widetilde{\pi}_Y(t) + 2\widetilde{\pi}_Y(0)$, where $\widetilde{\pi}_X(t) = \int_t^{+\infty} \overline{\widetilde{F}}_X(x)\,dx$ and $\widetilde{F}_X$ stands for the empirical distribution function of the sample associated with the variable X.

Figure 3 contains that representation. The values of t in the graphic cover all the values of the two samples.

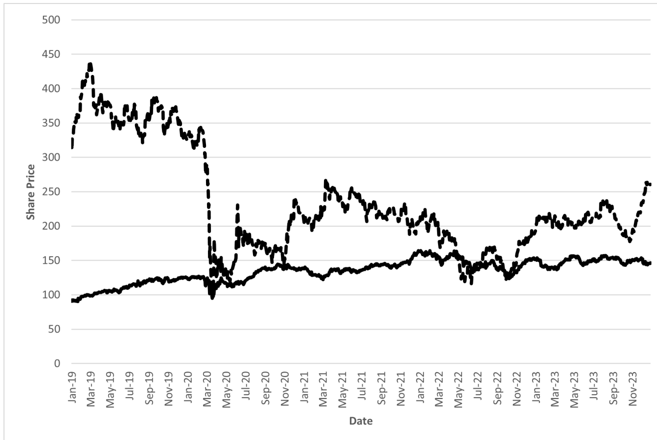

Figure 1. *Evolution of share prices of Boeing and P&G during the years 2019-23. Solid line for P&G, dashed line for Boeing. Dates in horizontal axis, prices in vertical axis.*

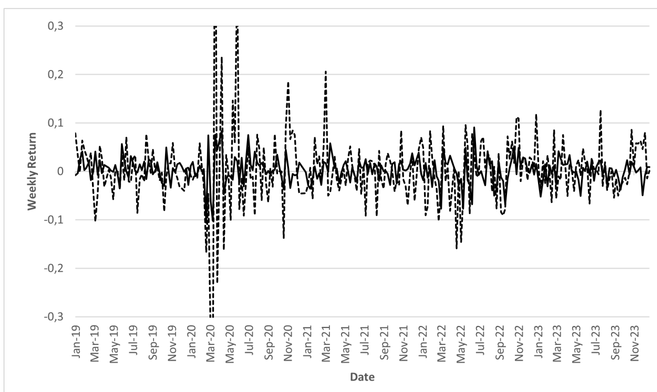

Figure 2. *Evolution of the weekly returns of Boeing and P&G during the years 2019-23. Solid line for P&G, dashed line for Boeing. Dates in horizontal axis, values of the weekly returns in vertical axis.*

Such a representation shows reasonable evidence that X is less that Y in the call ratio backspread stochastic order.

To draw a conclusion on such a relation, we state the following result.

Proposition 5.1. *Let X and Y be random variables with finite means. If $EX_+ \leq EY_+$ and $X_- \preceq_{icv} Y_-$, then $X \preceq_{crb}^0 Y$.*

Proof. The condition $X_- \preceq_{icv} Y_-$ is equivalent to

$$\int_{-\infty}^{t} F_{X_-}(x)\,dx \geq \int_{-\infty}^{t} F_{Y_-}(x)\,dx$$

for any $t \in \mathbb{R}$ (see Theorem 4.A.2 in Shaked and Shanthikumar (2007)). Notice that *a.e.* $F_{X_-}(x) = \overline{F}_X(-x)$ when $x \geq 0$, and $F_{X_-}(x) = 0$ if $x < 0$. Thus, when $t < 0$,

$$\int_{-\infty}^{t} F_{X_-}(x)\,dx = 0,$$

and if $t \geq 0$, we conclude that

$$\int_{-\infty}^{t} F_{X_-}(x)\,dx = \int_{0}^{t} F_{X_-}(x)\,dx = \int_{0}^{t} \overline{F}_X(-x)\,dx = \int_{-t}^{0} \overline{F}_X(x)\,dx.$$

Therefore, for any $t \geq 0$,

$$\int_{-t}^{0} \overline{F}_X(x)\,dx \geq \int_{-t}^{0} \overline{F}_Y(x)\,dx,$$

equivalently,

$$-\int_{t}^{0} \overline{F}_X(x)\,dx \leq -\int_{t}^{0} \overline{F}_Y(x)\,dx$$

for any $t < 0$. On the other hand, $EX_+ \leq EY_+$ is $\pi_X(0) \leq \pi_Y(0)$, hence,

$$-\int_{t}^{0} \overline{F}_X(x)\,dx + \pi_X(0) \leq -\int_{t}^{0} \overline{F}_Y(x)\,dx + \pi_Y(0)$$

for any $t < 0$, which is the same as $-\pi_X(t) + 2\pi_X(0) \leq -\pi_Y(t) + 2\pi_Y(0)$ for all $t < 0$, that is, $X \preceq_{crb}^{0} Y$. ■

Notice that the above result permits to use statistical inference techniques to test conditions which lead to the call ratio backspread stochastic order.

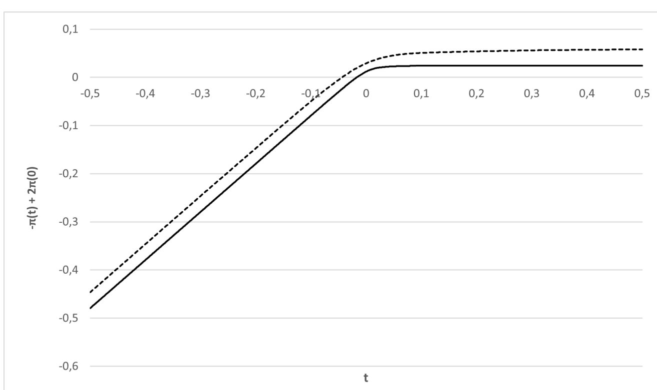

Figure 3. *Representation of the empirical version of the mappings* $t \to -\pi(t) + 2\pi(0)$ *with* $t \in (-0.5, 0.5)$*, solid line for P&G, dashed line for Boeing. Values of t in horizontal axis, values of the mappings in vertical axis.*

Some tests have been proposed to infer on the increasing convex order (equivalently, the increasing concave order), see for instance, Zardasht (2015), Berrendero and Cárcamo (2011), Scaillet and Topaloglou (2010) or Baringhaus and Grübel (2009). For our purpose, we took the CX10 test proposed by Berrendero and Cárcamo (2011) since it is quite intuitive, although other tests could be considered.

To apply the corresponding tests, we have divided at random the weekly returns of the period 2019-23 into two disjoint groups, one for Boeing and one for P&G, avoiding paired samples.

The classical run test was applied to all the involved samples (those of X_+, Y_+, X_- and Y_-). The corresponding p-values were greater than the usual level of significance 0.05. Thus, samples can be considered random, assumption needed for the application of the CX10 test and for the test on the comparison of the expectations of the positive parts of the variables. Moreover, the normality assumption was rejected for all the above variables.

Regarding the test with hypothesis

$$H_0 : X_- \preceq_{icv} Y_- \quad \text{versus} \quad H_1 : H_0 \textit{ is false},$$

the p-value of the corresponding samples was higher than 0.99. As a consequence, the null hypothesis is not rejected.

In relation to the test

$$H_0 : EX_+ \leq EY_+ \quad \text{versus} \quad H_1 : EX_+ > EY_+,$$

the p-value of the corresponding samples was 0.9956.

Making use of Proposition 5.1, we conclude that $X \preceq_{crb}^{0} Y$.

Observe that by Proposition 3.2, the relation $X \preceq_{crb}^{0} Y$ is equivalent to $X + p_2 \preceq_{crb}^{p_2} Y + p_2$ for any $p_2 \in \mathbb{R}$. Thus, if $p_2 = 1$, we obtain that

$$\frac{X_t}{X_{t-1}} \preceq_{crb}^{1} \frac{Y_t}{Y_{t-1}},$$

where X_t and Y_t stand for the weekly close prices of P&G and Boeing at week t, respectively.

That is, the expected benefit of a call ratio backspread derivative with asset the unit weekly revaluation of Boeing is greater (not lower) than the corresponding derivative with the asset unit weekly revaluation of P&G, whatever $p_1 < 1$. That shows that if for some $p_1 < 1$, the premium of the derivative associated with Boeing is lower than the premium of P&G, an arbitrage opportunity exists for those derivatives. Moreover, in case of equality of premiums, and investor should choose the Boeing derivative instead of the P&G option.

6. Final comments and conclusions

The present manuscript shows how the theory of stochastic orders can be used for reaching decisions on the allocations of funds in call ratio backspread derivatives. The mathematical model proposed in this article permits to compare investments in the above financial derivatives by means of a new family of stochastic orders. That allows to detect possible options of arbitrage.

This procedure entails some advantages with respect to other methods. The proposed technique does not require specific analytical equations or formulas of the prices of the assets, or particular probability distributions of those prices, like Brownian movements, or geometric Brownian movements. Moreover, an advantage of the new method is that when the order is satisfied, the expected benefits are ordered whatever price p_1. Thus, an investor does not need to attain some particular values of p_1 to be able to compare investments and find opportunities. Notice that when the order is satisfied and the premiums do not follow the same arrangement for a particular value of p_1, there exist arbitrage opportunities. On the other hand, we have proved the existence of supremum and infimum of two variables in the new orders, that brings advantages in optimization problems with stochastic constraints.

Acknowledgements

This work was supported by the Spanish Ministry of Science and Innovation under Grant MCI-20-PID2019-104486GB-I00 and Grant MCIU-22-PID2021-123461NB-C22, and Principado de Asturias Government under Grant AYUD/ 2021/50897.

References

Baringhaus, L. and Grübel, R. (2009). Nonparametric two-sample tests for increasing convex order. Bernoulli, 15, 99-123.

Belzunce, F., Martínez-Riquelme, C. and Mulero, J. (2016). An Introduction to Stochastic Orders. Elsevier/Academic Press, Amsterdam.

Berrendero, J.R. and Cárcamo, J. (2011). Tests for the second order stochastic dominance based on L-statistics. Journal of Business & Economic Statistics, 29, 260-270.

Cohen, G. (2005). The Bible of Options Strategies. Pearson Education, Inc., New Jersey.

Consigli, G., Dentcheva, D. and Maggioni, F. (2021). Stochastic optimization: theory and applications. Annals of Operational Research, 292, 575-580.

Dentcheva, D., Lai, B. and Ruszczyński, A. (2004). Dual methods for probabilistic optimization problems. Mathematical Methods of Operations Research, 60, 331-346.

Dentcheva, D. and Martínez, G. (2012). Two-stage stochastic optimization problems with stochastic ordering constraints on the recourse. European Journal of Operational Research, 219, 1-8.

Dentcheva, D. and and Wolfhagen, E. (2016). Two-stage optimization problems with multivariate stochastic order constraints. Mathematics of Operations Research, 41, 1-22.

Dixit, A.K. and Pindyck, R.S. (1994). Investment under Uncertainty. Princeton University Press, Princenton, New Jersey.

Hull, J.C. (2015). Options, futures and other derivatives. Pearson, Boston.

Jarrow, R. (1986). The relationship between arbitrage and first order stochastic dominance. Journal of Finance, 41, 915-921.

Levy, H. (2016). Stochastic dominance. Investment decision making under uncertainty. Third edition. Springer, Cham.

López-Díaz, M.C., López-Díaz, M. and Martínez-Fernández, S. (2018). Stochastic orders to approach investments in condor financial derivatives. Test, 27, 122-146.

Müller, A. (1997). Stochastic orders generated by integrals: a unified study. Advances in Applied Probability, 29, 414-428.

Müller, A. and Scarsini, M. (2006). Stochastic order relations and lattices of probability measures. SIAM Journal on Optimization, 16, 1024-1043.

Müller, A. and Stoyan, D. (2002). Comparison Methods for Stochastic Models and Risks. John Wiley & Sons, Chichester.

Ng, P., Wong, W.-K. and Xiao, Z. (2017). Stochastic dominance via quantile regression with applications to investigate arbitrage opportunity and market efficiency. European Journal of Operational Research, 261, 666-678.

Roberts, A.W. and Varberg, D.E. (1973). Convex functions. Pure and Applied Mathematics, Vol. 57. Academic Press, New York-London.

Scaillet, O. and Topaloglou, N. (2010). Testing for stochastic dominance efficiency. Journal of Business & Economic Statistics, 28, 169-180.

Shaked, M. and Shanthikumar, J.G. (2007). Stochastic Orders. Springer, New York.

Singh, A. and Selvamuthu, D. (2017). Mean-variance optimal trading problem subject to stochastic dominance constraints with second order autoregressive price dynamics. Mathematical Methods of Operations Research, 86, 29-69.

Zardasht, V. (2015). A test for the increasing convex order based on the cumulative residual entropy. Journal of the Korean Statistical Society, 44, 491-497.